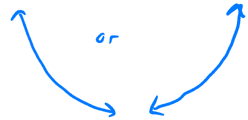
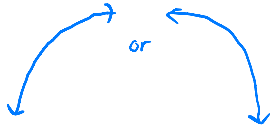
Lesson 18: Concavity and inflection points; Second derivative test.

Concativity

A function is **concave up** on (a, b) if $f''(x) > 0$ on (a, b)



A function is **concave down** on (a, b) if $f''(x) < 0$ on (a, b)



Inflection points

$x = c$ is an **inflection point** of $f(x)$ if the concavity of switches at $x = c$.

How to find inflection points

1) find all points such that $f''(x) = 0$ or DNE.
2) determine if the concavity switches near the points found in (1).

Examples

find the inflection points

① $y = (x^2 - 11x + 32)e^x$

$$y' = (2x - 11)e^x + (x^2 - 11x + 32)e^x$$
$$= e^x(x^2 - 9x + 21)$$
$$y'' = e^x(x^2 - 9x + 21) + e^x(2x - 9)$$
$$= e^x(x^2 - 7x + 12)$$

$y'' = 0$ or y'' DNE

never happens

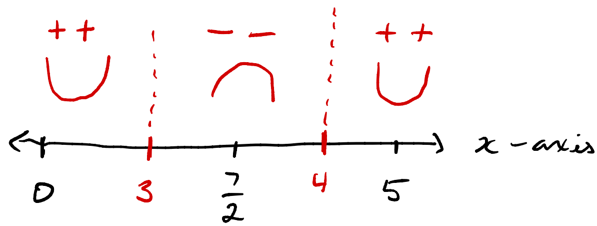
$e^x(x^2 - 7x + 12) = 0$

$x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$

$x = 3, 4$

f''



Concave up $(-\infty, 3)$ and $(4, \infty)$

Concave down on $(3,4)$.

$f''(0) = e^0(0^2 - 7(0) + 12) = 12$

$f''(7/2) = e^{7/2}\left(\left(\frac{7}{2}\right)^2 - 7\cdot\frac{7}{2} + 12\right) =$

$= e^{7/2}\left(\frac{49}{4} - \frac{49\cdot 2}{4} + \frac{48}{4}\right)$

$= e^{7/2}\left({}^{-1}/_4\right)$ negative!

$f''(5) = e^5(5^2 - 7(5) + 12)$

$= e^5(25 - 35 + 12)$

$= 2e^5$

3 and 4 are inflection points.

Second derivative test

Suppose $x = c$ is a critical point of $f(x)$.

1) If $f''(c) > 0$, then $x = c$ is relative min.

2) If $f''(c) < 0$, then $x = c$ is relative max.

Examples

(2) $y = 5 + 6x - 8x^3$ find relative extrema

$y' = 6 - 24x^2$

$y' = 0$ or y' DNE never happens

$6 - 24x^2 = 0$

$24x^2 = 6$

$x^2 = 1/4$

$x = \pm\sqrt{1/4} = \pm \frac{1}{2}$ critical points

$y'' = -48x$

$x = -\frac{1}{2}$ $\quad y''(-\frac{1}{2}) = -48(-\frac{1}{2}) = 24$

Second der. test $\Rightarrow$ $x = -\frac{1}{2}$ is rel. min

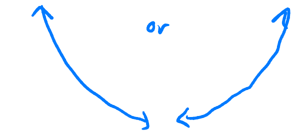
$x = \frac{1}{2}$ $\quad y''(\frac{1}{2}) = -48(\frac{1}{2}) = -24$

Second derivative test $\Rightarrow$ $x = \frac{1}{2}$ is rel. max.

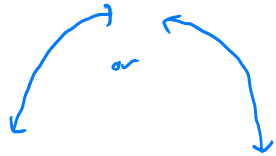
Lesson 18: Concavity and inflection points; Second derivative test.

Concavity

A function is **concave up** on (a, b) if
$f''(x) > 0$ on (a, b). $+ +$



A function is **concave down** on (a, b) if
$f''(x) < 0$ on (a, b). $- -$



Inflection point

$x = c$ is an **inflection point** of $f(x)$ if
the concavity switches at c.

How to find inflection points

1) find all points such that $f''(x) = 0$ or DNE

2) determine if the concavity switches near the points in (1). (f'' switches sign)

Examples

① $y = (x^2 - 11x + 32)e^x$ find inflection points.

$$y' = (2x - 11)e^x + (x^2 - 11x + 32)e^x$$
$$= e^x(x^2 - 9x + 21)$$

$$y'' = e^x(x^2 - 9x + 21) + e^x(2x - 9)$$
$$= e^x(x^2 - 7x + 12)$$

$y'' = 0$ or y'' DNE never happens

$$e^x(x^2 - 7x + 12) = 0$$

$$x^2 - 7x + 12 = 0$$

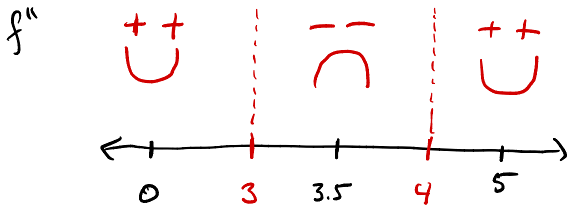
$$(x-4)(x-3) = 0$$

$$x = 3, 4$$

3, 4 inflection points

Concave up: $(-\infty, 3), (4, \infty)$

concave down: $(3, 4)$



$$f''(0) = e^0(0-4)(0-3) = 12$$

$$f''(3.5) = e^{3.5}(3.5-4)(3.5-3)$$
$$= e^{3.5}(-0.5)(0.5)$$
$$= -0.25e^{3.5}$$

$$f''(5) = e^5(5-4)(5-3)$$
$$= 2e^5$$

Second derivative test

Suppose $x=c$ is a critical point of $f(x)$.

1) If $f''(c) > 0$, then $x=c$ is a relative min.

2) If $f''(c) < 0$, then $x=c$ is a relative max.

② $y = 5 + 6x - 8x^3$ find relative extrema

$y' = 6 - 24x^2$

$0 = 6 - 24x^2$

$x = \pm 1/2$ ← Critical points

$y'' = -48x$

$y''(-\frac{1}{2}) = 24$

$-\frac{1}{2}$ is rel. min

$y''(+\frac{1}{2}) = -24$

$+\frac{1}{2}$ is rel. max.