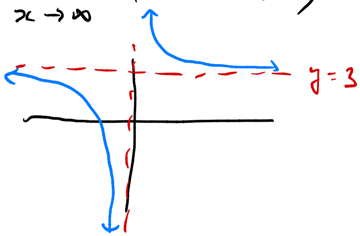
Lesson 21: Limits at infinity

$\lim_{x \to \infty} f(x)$: denote the value that f approaches as $x \to \infty$

$\lim_{x \to -\infty} f(x)$: denote the value that f approaches as $x \to -\infty$.

(1) $\lim_{x \to \infty} \left(3 + \frac{1}{x}\right) = 3$



x	10	100	1000	10000
$f(x)$	3.1	3.01	3.001	3.0001

$$\lim_{x \to \infty} \left(3 + \frac{1}{x}\right) = \lim_{x \to \infty} (3) + \lim_{x \to \infty} \left(\frac{1}{x}\right)$$

$$= 3 + 0 = 3$$

(2) $\lim_{x \to \infty} \left(\frac{10x + 6}{11x^2 + 20}\right) \approx \frac{10x}{11x^2} = \frac{10}{11x}$

$\lim_{x \to \infty} \left(\frac{10}{11x}\right) = 0$ so $\lim_{x \to \infty} \left(\frac{10x + 6}{11x^2 + 20}\right) = 0$

(3) $\lim_{x \to -\infty} \frac{3 - 7x^3}{x^3 + 2x^2 + 1} \sim \frac{-7x^3}{x^3} = -7$

$$\lim_{x \to -\infty} \left(\frac{3 - 7x^3}{x^3 + 2x^2 + 1}\right) = \lim_{x \to -\infty} (-7) = -7$$

(4) $\lim_{x \to -\infty} \left(\frac{7x^3 - 1}{20x^2 + x} \right) \sim \frac{7x^3}{20x^2} = \frac{7}{20} x \qquad \frac{7}{20}(-\infty) = -\infty$

$$\lim_{x \to -\infty} \left(\frac{7x^3 - 1}{20x^2 + x} \right) = \lim_{x \to -\infty} \left(\frac{7}{20} x \right) = -\infty$$

3 cases for rational functions as $x \to \pm \infty$

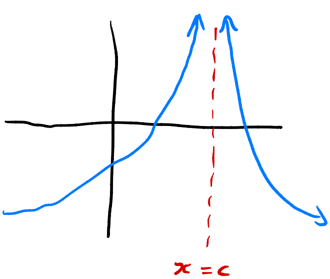
1) degree of top = degree of bottom
   Then limit is finite

2) degree of top > degree of bottom
   Then limit is $\pm \infty$

3) degree of top < degree of bottom
   Then limit is 0.

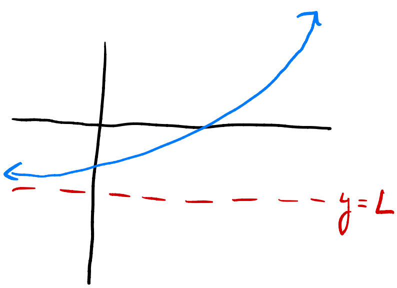
Asymptotes

$$y = \frac{f(x)}{g(x)}$$



Vertical

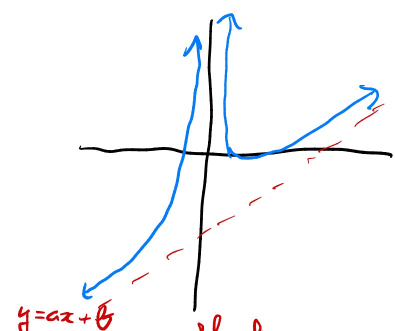
$g(c) = 0$



horizontal

$\lim_{x \to \infty} y = L$ or

$\lim_{x \to -\infty} y = L$



slant

$\deg f > \deg g$

$$y = ax + b + \frac{h(x)}{g(x)}$$

⑥ find vert/horz/slant asymptotes of

$$y = \frac{x^2+1}{x^2-25}$$

Vert: $0 = x^2 - 25 = (x-5)(x+5)$

$x = -5, \quad x = 5$

We have two verticle asy. $x = -5$, $x = 5$

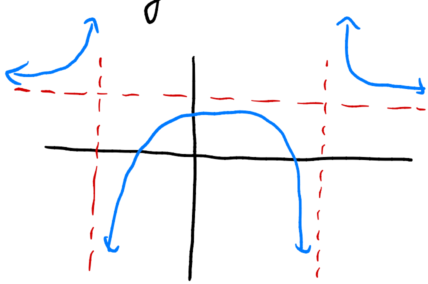
horz: $\lim_{x \to \infty} \frac{x^2+1}{x^2-25} \sim \frac{x^2}{x^2} = 1$

$$\left.\begin{aligned} \lim_{x \to \infty} \frac{x^2+1}{x^2-25} &= \lim_{x \to \infty} (1) = 1 \\ \lim_{x \to -\infty} \frac{x^2+1}{x^2-25} &= \lim_{x \to -\infty} (1) = 1 \end{aligned}\right\}$$ only need one of these

We have a horizontal asy. at $y = 1$

slant: $\deg(x^2+1) = 2$

$\deg(x^2-25) = 2$ so no slant asym.



⑦ find vert. / horz / slant asy. of

$$y = \frac{-4x^3 + 7x^2 + 22x + 28}{x^2 + 4}$$

$$\frac{-4x^3}{x^2} = -4x$$

$$\frac{7x^2}{x^2} = 7$$

$$\begin{array}{r} -4x + 7 \\ x^2 + 4 \overline{\big) -4x^3 + 7x^2 + 22x + 28} \\ -(-4x^3 - 16x) \\ \hline 7x^2 + 38x + 28 \\ -(7x^2 + 28) \\ \hline 38x \end{array}$$

$$\frac{-4x^3 + 7x^2 + 22x + 28}{x^2 + 4} = \underbrace{-4x + 7}_{\text{slant asy.}} + \frac{38x}{x^2 + 4}$$

$y = -4x + 7$ is slant asymt.

Vert: $x^2 + 4 = 0$ no sol.

so no vert.

horz: $\displaystyle\lim_{x \to \infty} \frac{-4x^3 + 7x^2 + 22x + 28}{x^2 + 4}$

$$= \lim_{x \to \infty} \left(-4x + 7 + \frac{38x}{x^2 + 4} \right) = -\infty$$

$$\lim_{x \to -\infty} \left(-4x + 7 + \frac{38x}{x^2 + 4} \right) = +\infty$$

no horz asy.

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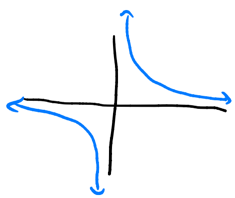
$\lim_{x \to -\infty} f(x)$: denote the value f approaches as $x \to -\infty$

(1) $\lim_{x \to \infty} \left(3 + \frac{1}{x}\right) = \lim_{x \to \infty} (3) + \lim_{x \to \infty} \left(\frac{1}{x}\right)$

$= 3 + 0 = 3$



x	10	100	1000	10,000
$\frac{1}{x}$	.1	.01	.001	.0001

(2) $\lim_{x \to \infty} \frac{10x + 6}{11x^2 + 20} \sim \frac{10x}{11x^2} = \frac{10}{11} \cdot \frac{1}{x} \to 0$ as $x \to \infty$

$\lim_{x \to \infty} \frac{10x + 6}{11x^2 + 20} = \lim_{x \to \infty} \left(\frac{10}{11} \cdot \frac{1}{x}\right) = 0$

(3) $\lim_{x \to -\infty} \frac{3 - 7x^3}{x^3 + 2x^2 + 1} \sim \frac{-7x^3}{x^3} = -7$

$\lim_{x \to -\infty} \frac{3 - 7x^3}{x^3 + 2x^2 + 1} = \lim_{x \to -\infty} (-7) = -7$

(4) $\lim\limits_{x \to -\infty} \dfrac{7x^3 - 1}{20x^2 + x} \sim \dfrac{7x^3}{20x^2} = \dfrac{7}{20} \cdot x$

$$\lim_{x \to -\infty} \frac{7x^3 - 1}{20x^2 + x} = \lim_{x \to -\infty} \left(\frac{7}{20} \cdot x \right) = -\infty$$

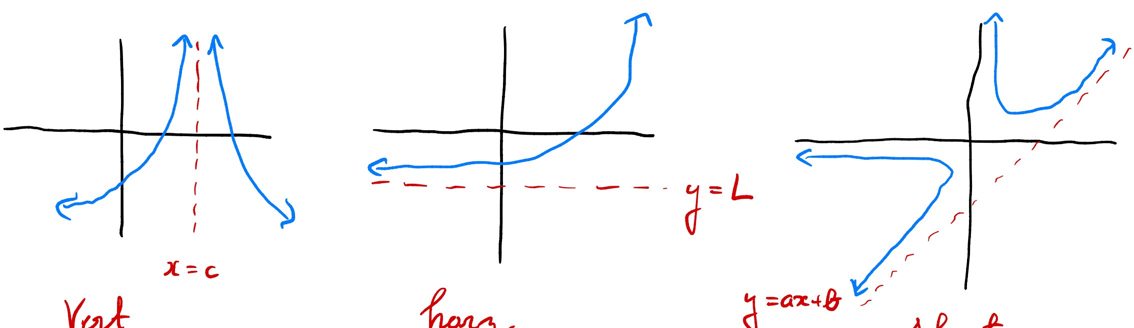
Cases for a rational function as $x \to \pm \infty$

1) deg of top > deg bottom, then
limit is $\pm \infty$

2) deg of top = deg bottom, then
limit is finite.

3) deg of top < deg of bottom, then
limit is 0.

Asymptotes $\quad y = f(x) / g(x)$



Vert. $\qquad$ horz. $\qquad$ slant.

Vert.: $g(c) = 0$

horz.: only need one $\left\{ \begin{array}{l} \lim\limits_{x \to \infty} y = L \quad \text{or} \\ \lim\limits_{x \to -\infty} y = L \end{array} \right.$

slant.: deg of f > deg of g

$$\frac{f(x)}{g(x)} = ax + b + \frac{h(x)}{g(x)}$$

(5) find vert / horz / slant asy. of $y = \dfrac{x^2+1}{x^2-25}$

vert: $0 = x^2 - 25$

$= (x-5)(x+5)$

$x = -5, \quad x = 5$

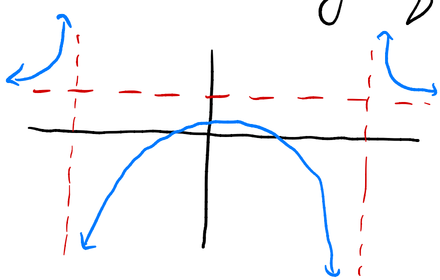
Verticle asy. at $x = -5, 5$

horz: $\lim_{x \to \infty} \dfrac{x^2+1}{x^2-25} \sim \dfrac{x^2}{x^2} = 1$

$\lim_{x \to \infty} \dfrac{x^2+1}{x^2-25} = 1, \quad \lim_{x \to -\infty} \dfrac{x^2+1}{x^2-25} = 1$

We have horz asy. at $y = 1$

slant: Since deg of the top is not greater than the deg of bottom, no slant asy.



(6) find vert / horz / slant asy. of

$$y = \frac{-4x^3 + 7x^2 + 22x + 28}{x^2 + 4}$$

* make sure your terms are in descending order. *

$$\begin{array}{r l}
 & -4x + 7 \\
x^2 + 4 \;\big) & -4x^3 + 7x^2 + 22x + 28 \\
 & -(-4x^3 - 16x) \\
\hline
 & 7x^2 + 38x + 28 \\
 & -(7x^2 + 28) \\
\hline
 & 38x
\end{array}$$

$38x$ ← remainder

$$\frac{-4x^3}{x^2} = -4x$$

$$\frac{7x^2}{x^2} = 7$$

$$\frac{38x}{x^2} = \cancel{\frac{38}{x}}$$

$$\frac{-4x^3 + 7x^2 + 22x + 28}{x^2 + 4} = \underbrace{-4x + 7}_{\text{slant}} + \frac{38x}{x^2 + 4}$$

$y = -4x + 7$ is the slant asy.

Vert: $x^2 + 4 = 0$

$x^2 = -4$ no solutions,

no verticle asy.

horz: $\dfrac{-4x^3+7x^2+22x+28}{x^2+4} = -4x+7 + \dfrac{38x}{x^2+4}$

$$\lim_{x\to\infty}\left(-4x+7+\frac{38x}{x^2+4}\right)$$

$$= \lim_{x\to\infty}(-4x+7) + \lim_{x\to\infty}\left(\frac{38x}{x^2+4}\right)$$

$$= -\infty + 0$$

$$\lim_{x\to-\infty}\left(-4x+7+\frac{38x}{x^2+4}\right) = +\infty$$

so no horz asy.