Lesson 31: Fundamental Theorem of Calculus I

Recall: Let $f(x)$ be a function with an antiderivative $F(x)$.

Indefinite integral: $\int f(x)\,dx = F(x) + C$

Definite integral: $\int_a^b f(x)\,dx =$ Signed area under $f(x)$ on $[a, b]$

Fundamental Theorem of Calculus (FTC)

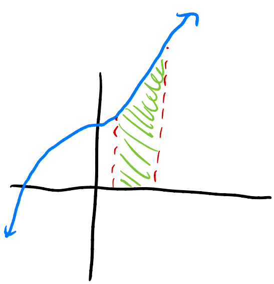
$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \left(=: F(x)\Big|_a^b \right)$$

① Evaluate $\int_1^2 36x^3 + 9\,dx$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

First need an antiderivative.

$$\int_1^2 36x^3 + 9\,dx = 36\int_1^2 x^3\,dx + 9\int_1^2 dx$$



$$= 36 \cdot \frac{x^4}{4}\Big|_1^2 + 9x\Big|_1^2$$

$$= 36\left(\frac{(2)^4}{4} - \frac{(1)^4}{4}\right) + 9(2-1)$$

$$= 36\left(\frac{16}{4} - \frac{1}{4}\right) + 9(1)$$

$$= 9(15) + 9 = 144$$

(2) Evaluate $\int_0^6 3e^x + 9\,dx$

$$\int_0^6 3e^x + 9\,dx = 3\int_0^6 e^x\,dx + 9\int_0^6 dx$$

$$= 3e^x \Big|_0^6 + 9x \Big|_0^6$$

$$= 3(e^6 - e^0) + 9(6 - 0)$$

$$= 3(e^6 - 1) + 54$$

$$= 3e^6 + 51 \checkmark$$

(3) Evaluate $\int_1^4 \frac{x^2 + x^3}{\sqrt{x}}\,dx$

$$\frac{x^2 + x^3}{\sqrt{x}} = \frac{x^2 + x^3}{x^{1/2}} = \frac{x^{2 = \frac{4}{2}}}{x^{1/2}} + \frac{x^{3 = \frac{6}{2}}}{x^{1/2}} = x^{3/2} + x^{5/2}$$

$$\int_1^4 x^{3/2} + x^{5/2}\,dx = \int_1^4 x^{3/2}\,dx + \int_1^4 x^{5/2}\,dx$$

$$= \frac{2}{5}x^{5/2}\Big|_1^4 + \frac{2}{7}x^{7/2}\Big|_1^4$$

$$= \frac{2}{5}\left(4^{5/2} - 1^{5/2}\right) + \frac{2}{7}\left(4^{7/2} - 1^{7/2}\right)$$

$$= \frac{2}{5}\left(2^5 - 1\right) + \frac{2}{7}\left(2^7 - 1\right)$$

$$= \frac{2^6}{5} - \frac{2}{5} + \frac{2^8}{7} - \frac{2}{7}$$

$$= \frac{2^6 \cdot 7 - 2 \cdot 7 + 2^8 \cdot 5 - 2 \cdot 5}{35}$$

$$= \frac{1704}{35}$$

(4) Evaluate $\int_{-\pi/3}^{\pi/3} 9 \tan x \cos x + 9 \, dx$.

$$\tan x \cos x = \frac{\sin x}{\cos x} \cdot \cos x = \sin x$$

$$\int_{-\pi/3}^{\pi/3} 9 \sin x + 9 \, dx = 9\int_{-\pi/3}^{\pi/3} \sin x \, dx + 9\int_{-\pi/3}^{\pi/3} dx$$

$$= -9 \cos x \Big|_{-\pi/3}^{\pi/3} + 9x \Big|_{-\pi/3}^{\pi/3}$$

$$= -9\left(\cos\left(\frac{\pi}{3}\right) - \cos\left(-\frac{\pi}{3}\right)\right) + 9\left(\frac{\pi}{3} - \frac{-\pi}{3}\right)$$

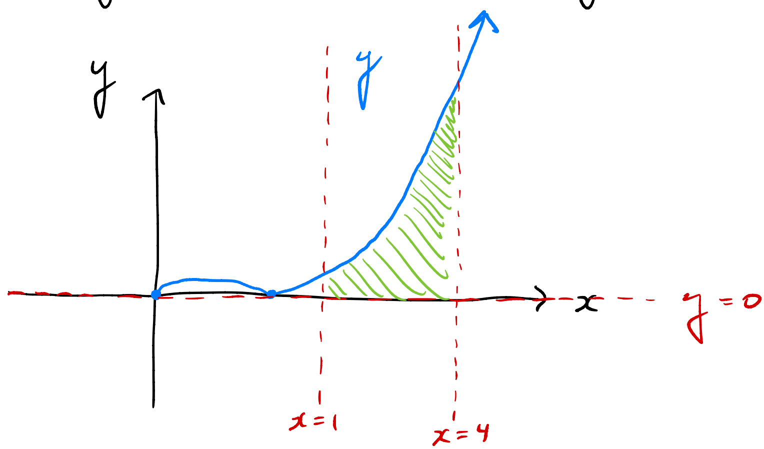
$$= -9\left(\frac{1}{2} - \frac{1}{2}\right) + 9\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= -9(0) + 9 \cdot \frac{2\pi}{3}$$

$$= 6\pi$$

5 Find the <u>area</u> enclosed by the graphs

$$y = 6\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2, \quad y = 0, \quad x = 1, \quad x = 4$$



$$\int_1^4 6\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2 dx$$

$$\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2 = \left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)$$

$$= \frac{x}{25} - \frac{x^{3/2}}{15} - \frac{x^{3/2}}{15} + \frac{x^2}{9}$$

$$= \frac{x}{25} - \frac{2x^{3/2}}{15} + \frac{x^2}{9}$$

$$6\int_1^4 \frac{x}{25} - \frac{2x^{3/2}}{15} + \frac{x^2}{9}\,dx$$

$$= 6\left[\frac{1}{25}\int_1^4 x\,dx - \frac{2}{15}\int_1^4 x^{3/2}\,dx + \frac{1}{9}\int_1^4 x^2\,dx\right]$$

$$= 6\left[\frac{1}{25}\cdot\frac{1}{2}x^2\Big|_1^4 - \frac{2}{15}\cdot\frac{2}{5}x^{5/2}\Big|_1^4 + \frac{1}{9}\cdot\frac{1}{3}x^3\Big|_1^4\right]$$

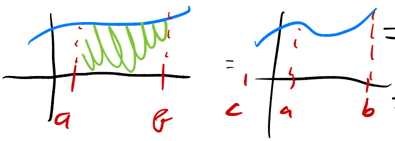
$$= 6\left[\frac{1}{50}\left(4^2 - 1^2\right) - \frac{4}{75}\left(4^{5/2} - 1^{5/2}\right) + \frac{1}{27}\left(4^3 - 1^3\right)\right]$$

HW 30 #6

$$\int_{-9}^{4} f(x)\,dx = -4, \quad \int_{-6}^{-11} f(x)\,dx = 0$$

$$\int_{-11}^{4} f(x)\,dx = -13$$

a) $$\int_{-6}^{4} f(x)\,dx = \int_{-6}^{-11} f(x)\,dx + \int_{-11}^{4} f(x)\,dx$$



$= 0 + -13$

$= -13$

b) $$\int_{4}^{-9} f(x)\,dx = -\int_{-9}^{4} f(x)\,dx = -(-4) = 4$$

Lesson 31: Fundamental Theorem of Calculus I

Recall: Let $f(x)$ be a function with <u>an</u> antiderivative $F(x)$, then

Indefinite integral: $\int f(x)\,dx := F(x) + C$

Definite integral: $\int_a^b f(x)\,dx :=$ signed area under $f(x)$ on $[a, b]$

Fundamental Theorem of Calculus (FTC)

$$\int_a^b f(x)\,dx = F(b) - F(a) =: F(x)\Big|_a^b$$

$$\cancel{+C} \qquad \cancel{-C}$$

① Evaluate $\int_1^2 36x^3 + 9\,dx$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$n \neq -1$$

$$= 36\int_1^2 x^3\,dx + 9\int_1^2 dx$$

$$= 36 \cdot \frac{1}{4}x^4 \Big|_1^2 + 9x\Big|_1^2$$

$$= 9(2^4 - 1^4) + 9(2-1)$$

$$= 9 \cdot 15 + 9 = 144$$

(2) Evaluate $\int_0^6 3e^x + 9 \, dx$

$$= 3\int_0^6 e^x \, dx + 9\int_0^6 dx$$

$$= 3 e^x \Big|_0^6 + 9x \Big|_0^6$$

$$= 3(e^6 - e^0) + 9(6 - 0)$$

$$= 3(e^6 - 1) + 54$$

$$= 3e^6 + 51 \quad \checkmark$$

(3) Evaluate $\int_{-\pi/3}^{\pi/3} 9 \tan x \cos x + 9 \, dx$

$$\tan x \cos x = \frac{\sin x}{\cos x} \cdot \cos x = \sin x$$

$$= \int_{-\pi/3}^{\pi/3} 9 \sin x + 9 \, dx = 9\int_{-\pi/3}^{\pi/3} \sin x \, dx + 9\int_{-\pi/3}^{\pi/3} dx$$

$$= -9 \cos x \Big|_{-\pi/3}^{\pi/3} + 9x \Big|_{-\pi/3}^{\pi/3}$$

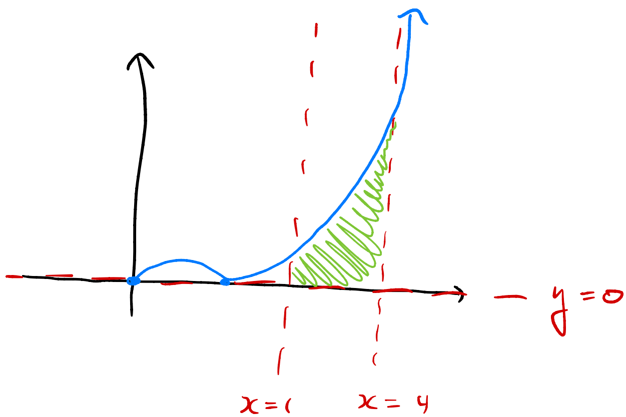
$$= -9\left(\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{-\pi}{3}\right)\right) + 9\left(\frac{\pi}{3} - \frac{-\pi}{3}\right)$$

$$= -9\left(\frac{1}{2} - \frac{1}{2}\right) + 9\left(\frac{2\pi}{3}\right)$$

$$= 6\pi$$

(4) Find the area enclosed by the graphs

$$y = 6\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2, \quad y = 0, \quad x = 1, \quad x = 4$$



$$\int_1^4 6\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2 dx$$

$$\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2 = \left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)$$

$$= \frac{x}{25} - \frac{x^{3/2}}{15} - \frac{x^{3/2}}{15} + \frac{x^2}{9}$$

$$= \frac{x}{25} - \frac{2x^{3/2}}{15} + \frac{x^2}{9}$$

$$6\int_1^4 \frac{x}{25} - \frac{2x^{3/2}}{15} + \frac{x^2}{9}\,dx$$

$$= 6\left[\frac{1}{25}\cdot\frac{1}{2}x^2 - \frac{2}{15}\cdot\frac{2}{5}x^{5/2} + \frac{1}{9}\cdot\frac{1}{3}x^3\right]\Bigg|_1^4$$

$$= 6\left[\frac{1}{50}x^2 - \frac{4}{75}x^{5/2} + \frac{1}{27}x^3\right]\Bigg|_1^4$$

$$= 6\left[\frac{1}{50}(4)^2 - \frac{4}{75}(4)^{5/2} + \frac{1}{27}(4)^3\right.$$

$$\left. - \left(\frac{1}{50}(1)^2 - \frac{4}{75}(1)^{5/2} + \frac{1}{27}(1)^3\right)\right]$$

$$= 6\left[\frac{16}{50} - \frac{4\cdot 2^5}{75} + \frac{2^6}{27} - \frac{1}{50} + \frac{4}{75} - \frac{1}{27}\right]$$

$$= 6\left[\frac{15}{50} - \frac{124}{75} + \frac{63}{27}\right]$$