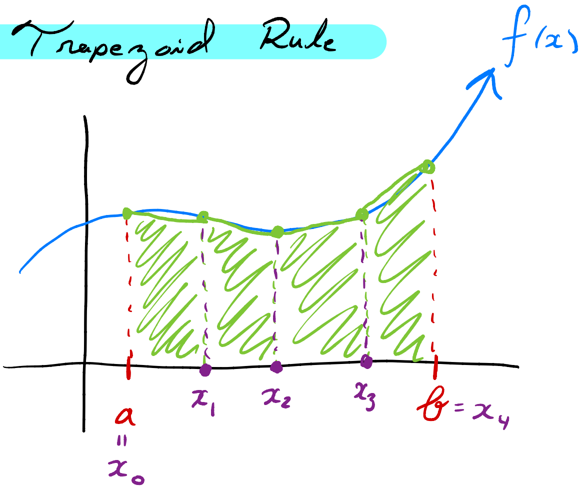
Lesson 33: Numerical integration

Recall: We computed $\int_a^b f(x)\,dx$ using the FTC, we need an antiderivative for $f(x)$!
In practice this is hard (sometimes impossible).

Trapezoid Rule



$$\Delta x = \frac{b-a}{n}$$

$n = \#$ of trapezoids ($n=4$)

$$x_i = a + i\Delta x$$

$$\int_a^b f(x)\,dx \approx \frac{f(x_0)+f(x_1)}{2}\Delta x + \frac{f(x_1)+f(x_2)}{2}\Delta x + \frac{f(x_2)+f(x_3)}{2}\Delta x + \frac{f(x_3)+f(x_4)}{2}\Delta x$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

In general for n trapezoids,

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$$

$$\int_a^b f(x)\,dx \approx T_n := \frac{\Delta x}{2}\Big[f(x_0) + \underbrace{2f(x_1) + \cdots + 2f(x_{n-1})}_{\text{multiplied by 2}} + f(x_n)\Big]$$

Examples

① Approximate $\int_3^5 7x^2+1\,dx$ using the trapezoid rule with $n=4$.

$$f(x) = 7x^2+1 \qquad \Delta x = \frac{5-3}{4} = \frac{2}{4} = 1/2$$

$$\int_3^5 7x^2+1\,dx \approx T_4 = \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\Big]$$

$$x_i = a + i\Delta x = 3 + i\cdot\frac{1}{2}$$

$x_0 = 3 + 0\cdot\frac{1}{2} = 3$

$x_1 = 3 + 1\cdot\frac{1}{2} = 7/2$

$x_2 = 3 + 2\cdot\frac{1}{2} = 4$

$x_3 = 3 + 3\cdot\frac{1}{2} = 9/2$

$x_4 = 3 + 4\cdot\frac{1}{2} = 5$

$$T_4 = \frac{1/2}{2}\Big[7(3)^2+1 + 2\left(7\left(\tfrac{7}{2}\right)^2+1\right) + 2\left(7(4)^2+1\right)$$

$$+ 2\left(7\left(\tfrac{9}{2}\right)^2+1\right) + 7(5)^2+1\Big]$$

(2) Approximate $\int_{-2}^{2} e^{x^2}\,dx$ using the trapezoid rule with $n=4$.

$f(x) = e^{x^2}$ $\qquad \Delta x = \frac{2-(-2)}{4} = \frac{4}{4} = 1$

$$\int_{-2}^{2} e^{x^2}\,dx \approx T_4 = \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2)$$
$$+ 2f(x_3) + f(x_4)\Big]$$

$x_i = a + i\Delta x = -2 + i$

$x_0 = -2 + 0 = -2$ $\qquad x_3 = -2 + 3 = 1$

$x_1 = -2 + 1 = -1$ $\qquad x_4 = -2 + 4 = 2$

$x_2 = -2 + 2 = 0$

$$T_4 = \frac{1}{2}\Big[e^{(-2)^2} + 2e^{(-1)^2} + 2e^{(0)^2} + 2e^{(1)^2} + e^{(2)^2}\Big]$$

$$= \frac{1}{2}\Big[e^4 + 2e + 2 + 2e + e^4\Big]$$

$$= \frac{1}{2}\Big[2e^4 + 4e + 2\Big] = e^4 + 2e + 1$$

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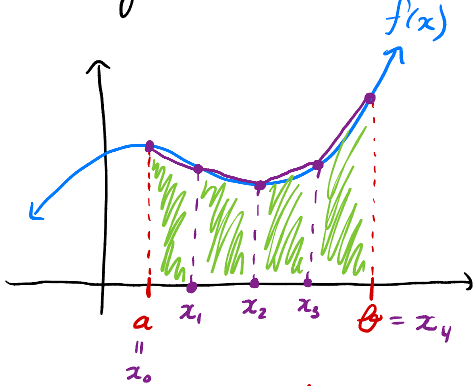
Trapezoid rule



$$\Delta x = \frac{b-a}{n}$$ n = # of trapezoids $(n=4)$

$$x_i = a + i\Delta x$$

$$\text{Area of trapezoid} = \frac{b_1 + b_2}{2} h$$

$$\int_a^b f(x)\,dx \approx \frac{f(x_0)+f(x_1)}{2}\Delta x + \frac{f(x_1)+f(x_2)}{2}\Delta x + \frac{f(x_2)+f(x_3)}{2}\Delta x + \frac{f(x_3)+f(x_4)}{2}\Delta x$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right]$$

In general for n trapezoids,

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$$

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Examples

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$$f(x) = 7x^2 + 1 \qquad \Delta x = \frac{5-3}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\int_3^5 7x^2 + 1\,dx \approx T_4$$

$$= \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\Big]$$

$$x_i = a + i\Delta x = 3 + i\left(\tfrac{1}{2}\right)$$

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$$\left.2\left(7\left(\tfrac{9}{2}\right)^2+1\right)+7(5)^2+1\right]$$

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