Lesson 34: Exponential growth

Given the differential equation $\frac{dy}{dt} = 2y$, how do we solve for y?

By guess and check!

What functions derivative is just itself?

guess: $y = e^t$ $\qquad \frac{dy}{dt} = e^t = y$

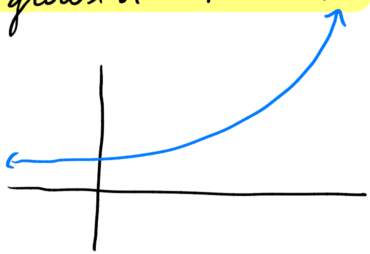
guess: $y = e^{2t}$ $\qquad \frac{dy}{dt} = 2e^{2t} = 2y$

In fact all solutions to $\frac{dy}{dt} = 2y$ are of the form $\underline{y = ce^{2t}}$ for some constant c.

In general, given the ODE $\frac{dy}{dt} = ky$ all solutions are of the form $y = ce^{kt}$

c is called the initial value

k is called the proportionality const. or growth rate if k and c are both greater than zero, then we say $y = ce^{kt}$ is an exponential growth model.



$c > 0$, $k > 0$

Examples

(1) A population of bacteria, $P(t)$, where t is time in days, is growing proportional to the population itself and the growth rate is 0.2.

If the initial population is 40, what is the population after 40 days?

$$\frac{dP}{dt} = kP = (0.2)P$$

$$P(t) = ce^{0.2t} \qquad 40 = P(0) = ce^{0.2(0)} = c$$

$$P(t) = 40e^{0.2t}$$

$$P(40) = 40e^{\frac{2}{10}\cdot 40} = 40e^{8} \approx 119238.32 \text{ bacteria}$$

(2) The rate of change of a population of a town is $\underline{\frac{dP}{dt} = kt}$, where t in years.

If at $t = 3$ $P = 30{,}000$
$t = 5$ $P = 40{,}000$, then

what is <u>the population when $t = 10$</u>?

$P(t) = ce^{kt}$, $P(10) = ??$

$$30{,}000 = P(3) = ce^{3k}$$
$$40{,}000 = P(5) = ce^{5k}$$

$$40{,}000 = ce^{5k}$$
$$30{,}000 = ce^{3k}$$

divide first eq. by second eq.

$$\frac{4}{3} = \frac{40{,}000}{30{,}000} = \frac{ce^{5k}}{ce^{3k}} = e^{5k-3k} = e^{2k}$$

$$\frac{4}{3} = e^{2k}$$

$$\ln \frac{4}{3} = \ln\left(e^{2k}\right) = 2k$$

$$k = \frac{1}{2} \ln \frac{4}{3} = \ln \sqrt{\frac{4}{3}} = \ln \frac{2}{\sqrt{3}}$$

$$30{,}000 = ce^{3 \ln 2/\sqrt{3}}$$
$$= ce^{\ln 8/3\sqrt{3}}$$
$$= c\left(\frac{8}{3\sqrt{3}}\right)$$

$$c = \frac{30,000}{8} \cdot 3\sqrt{3}$$

$$P(10) = ce^{kt} = \frac{30,000}{8} \cdot 3\sqrt{3} \cdot e^{10 \cdot \ln 2/\sqrt{3}}$$

$$\approx 82112 \text{ people}$$

③ Jessica deposited \$40,000 into a savings account with interest compounded continuously. The annual rate of interest is 3%. How much money does she have in the account 12 years from now?

P = ammount of money in the account

t = time in years

$P = ce^{kt}$ $\qquad k = 0.03 \leftarrow$ convert the percent to decimal

$$40,000 = P(0) = ce^{0.03 \cdot 0} = c$$

$$P(t) = 40,000\, e^{0.03t}$$

$$P(12) = 40,000\, e^{0.03 \cdot 12} \approx \$57333.18$$

(4) Shiori deposited \$700 into her savings account in which interest is compounded continuously. After 20 years she has \$1750 in the account.

What is the annual interest rate?

$P(t) = ce^{kt}$ $\quad$ P = \$ in account

t = time in years

$$700 = P(0) = ce^{k(0)} = c$$

$$1750 = P(20) = 700e^{k(20)}$$

$$\frac{1750}{700} = e^{20k}$$

$$\ln \frac{1750}{700} = 20k$$

$$k = \frac{1}{20} \ln \frac{1750}{700} \approx 0.05$$

Annual interest rate is 5% ✓

HW 33 #5

$$\int_{-1.4}^{0.4} \frac{\tan x}{x+5}\,dx \qquad n = 3$$

$$b = 0.4 \qquad a = -1.4 \qquad \Delta x = \frac{b-a}{n} = \frac{0.4-(-1.4)}{3} = 0.6$$

$$f(x) = \frac{\tan x}{x+5}$$

$$T_3 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3) \right]$$

$$x_i = a + i\Delta x = -1.4 + i(0.6)$$

$$x_0 = -1.4 \qquad x_3 = 0.4$$

$$x_1 = -0.8$$

$$x_2 = -0.2$$

$$T_3 = \frac{0.6}{2}\left[\frac{\tan(-1.4)}{-1.4+5} + 2\frac{\tan(-0.8)}{-0.8+5} + 2\frac{\tan(-0.2)}{-0.2+5} + \frac{\tan(0.4)}{0.4+5} \right]$$

Make sure your calculator is in radians!

Lesson 34: Exponential growth

Given the differential equation $\frac{dy}{dt} = 2y$, then how do we solve for y?

By guess and check!

What function's derivative includes just itself?

guess 1: $y = e^t$

$$\frac{dy}{dt} = e^t = y \quad \times$$

guess 2: $y = e^{2t}$

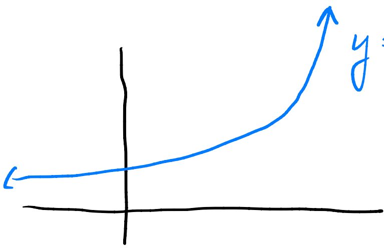
$$\frac{dy}{dt} = 2e^{2t} = 2y \quad \checkmark$$

In fact all solutions are of the form $y = ce^{2t}$ for some constant c.

In general given the ODE $\frac{dy}{dt} = ky$ all solutions are of the form $y = ce^{kt}$ where c is some constant.

c is called the initial value of y

k is called the proportionality constant or growth rate if c and k are both greater than zero, then we say $y = ce^{kt}$ is an exponential growth model.



Examples

① A population of bacteria, $P(t)$, where t is time in days, is growing proportional to the population itself and the growth rate is 0.2. If the initial population is 40, what is the population after 40 days?

$P(t) = ce^{kt}$, $k = 0.2$

$$40 = P(0) = ce^{(0.2)\cdot 0} = c$$

$$c = 40$$

$$P(t) = 40e^{0.2t}$$

$$P(40) = 40e^{0.2 \cdot 40} \approx 119238.32$$

bacteria

(2) The rate of change of a population of a town is $\frac{dP}{dt} = kt$, where t is time in years.

If at $t = 3$ $P = 30,000$ and $t = 5$ $P = 40,000$, then what is the population when $t = 10$?

$$P(t) = ce^{kt} \qquad P(10) = ??$$

$$30,000 = P(3) = ce^{3k}$$
$$40,000 = P(5) = ce^{5k}$$

$$40,000 = ce^{5k}$$
$$30,000 = ce^{3k}$$

divide the first eq. by the second eq.

$$\frac{4}{3} = \frac{40,000}{30,000} = \frac{ce^{5k}}{ce^{3k}} = e^{5k-3k} = e^{2k}$$

$$\frac{4}{3} = e^{2k}$$

$$\ln\frac{4}{3} = \ln(e^{2k}) = 2k$$

$$k = \frac{1}{2}\ln\frac{4}{3} = \ln\sqrt{\frac{4}{3}} = \ln\frac{2}{\sqrt{3}}$$

$$30{,}000 = c\,e^{\ln(2/\sqrt{3})\cdot 3}$$

$P(3) = 30{,}000$

$$= c\,e^{\ln(2/\sqrt{3})^3}$$

$$= c\,e^{\ln(8/3\sqrt{3})}$$

$$= c\,\frac{8}{3\sqrt{3}}$$

$$c = \frac{30{,}000}{8}\cdot 3\sqrt{3}$$

$$P(10) = \frac{30{,}000}{8}\cdot 3\sqrt{3}\;e^{(\ln 2/\sqrt{3})\cdot 10}$$

≈ 82112 people after 10 years

(3) Jessica deposited \$40,000 into a savings account with interest compounded continuously.

The annual rate of interest is 3%.

How much money does she have in the account 12 years from now?

P = \$ in the account

t = time in years

$k = 0.03$

Convert % to decimal

$c = 40{,}000$

$$P(t) = c\,e^{kt} = 40{,}000\,e^{0.03t}$$

$$P(12) = 40{,}000\, e^{0.03(12)} \approx \$5733.18\ .$$