Lesson 35: Exponential decay

Exponential decay model

$\frac{dy}{dt} = ky$ the solutions are of the form

$$y = ce^{kt}$$

if k is negative, then this is called an exponential decay model.

Examples

① A population of fish, $P(t)$, is decreasing at a rate proportional to itself, t in years.

If $P = 100{,}000$ when $t = 2$ and $P = 50{,}000$ when $t = 5$, then what is the population of fish when $t = 10$?

$$\frac{dP}{dt} = kP \Rightarrow P(t) = ce^{kt} \quad , \quad P(10) = ??$$

$$100{,}000 = P(2) = ce^{2k}$$

$$50{,}000 = P(5) = ce^{5k}$$

$$50{,}000 = ce^{5k}$$

$$100{,}000 = ce^{2k}$$

divide eq. 1 by eq. 2

$$\frac{1}{2} = \frac{50{,}000}{100{,}000} = \frac{ce^{5k}}{ce^{2k}} = e^{5k-2k} = e^{3k}$$

$$\frac{1}{2} = e^{3k}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{3k}\right)$$

$$\cancel{\ln 1} - \ln 2 = 3k$$

$$0 - \ln 2 = 3k$$

$$k = -\frac{1}{3}\ln 2 < 0$$ so exponential decay!

$$50{,}000 = c\, e^{5\left(-\frac{1}{3}\ln 2\right)}$$

$$c = \frac{50{,}000}{e^{-\frac{5}{3}\ln 2}}$$

$$= \frac{50{,}000}{e^{\ln 2^{-5/3}}}$$

$$= \frac{50{,}000}{2^{-5/3}}$$

$$P(10) = c\,e^{k(10)} = \frac{50{,}000}{2^{-5/3}}\, e^{\left(-\frac{1}{3}\ln 2\right)(10)}$$

$$\approx 15{,}749.01$$ fish after 10 years.

(2) The radio active isotope ^{226}Ra has a half life of about 1599 years.
There are 85g of ^{226}Ra now.
How much remains after 1400 years?

half life $\Rightarrow$ $P(t) = ce^{kt}$

P = ammount of ^{226}Ra

t = time in years.

$P(1400) = ??$

$$85 = P(0) - ce^{k(0)} = ce^{0} = c$$

$$P(t) = 85e^{kt}$$

$$\frac{85}{2} = P(1599) = 85e^{k(1599)}$$

$$\frac{1}{2} = e^{1599k}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{1599k}\right)$$

$$-\ln 2 = 1599k$$

$$k = \frac{-\ln 2}{1599}$$

$$P(1400) = 85e^{-\frac{\ln 2}{1599}\cdot 1400}$$

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$$y = ce^{kt}$$

if k is negative, then this is called an exponential decay model.

Examples

1. The radio active isotope ^{226}Ra has a half life of about 1599 years. There are 85 g of ^{226}Ra now. How much remains after 1400 years?

half life $\Longrightarrow$ $P(t) = ce^{kt}$

P: ammount of ^{226}Ra

t: time in years

$$P(1400) = ??$$

$$85 = P(0) = ce^{k(0)} = ce^{0} = c$$

$$P(t) = 85e^{kt}$$

$$\frac{85}{2} = P(1599) = 85e^{k(1599)}$$

$$\frac{1}{2} = e^{1599k}$$

$$\ln\left(\tfrac{1}{2}\right) = \ln\left(e^{1599k}\right)$$

$$\cancel{\ln 1} - \ln 2 = 1599k$$

$$0 \quad -\ln 2 = 1599k$$

$$k = \frac{-\ln 2}{1599} < 0$$ exponential decay

$$P(1400) = 85e^{\frac{-\ln 2}{1599}\cdot 1400}$$

$$\approx 46.33 \text{ g of } {}^{226}Ra$$

(2) The radioactive isotope ${}^{239}Pu$ has a half life of about 24100 years.
After 1500 years there are 5g of ${}^{239}Pu$.

a) What is the initial quantity of ${}^{239}Pu$?

What is $c = ??$

$P(t) = ce^{kt}$ where P: ammount of ${}^{239}Pu$
t: time in years.

$$\frac{c}{2} = P(24100) = ce^{k(24100)}$$

$$\frac{1}{2} = e^{24100k}$$

$$\ln\left(\tfrac{1}{2}\right) = \ln\left(e^{24100k}\right)$$

$$-\ln 2 = 24100k$$

$$k = \frac{-\ln 2}{24100} < 0$$

$$P(t) = ce^{\frac{-\ln 2}{24100}t}$$

$$5 = P(1500) = ce^{\frac{-\ln 2}{24100}\cdot 1500}$$

$$c = \frac{5}{e^{-\frac{1500}{24100}\cdot \ln 2}}$$

$$= 5e^{\frac{15}{241}\ln 2}$$

$$= 5e^{\ln 2^{15/241}}$$

$$= 5 \cdot 2^{15/241}$$

≈ 5.22 grams of ^{239}Pu initially