

Lesson 35: Exponential decay

Exponential decay model

$$\frac{dy}{dt} = ky \quad \text{the solutions are of the form}$$
$$y = ce^{kt}$$

if k is negative, then this is called an exponential decay model.

Examples

① A population of fish, $P(t)$, is decreasing at a rate proportional to itself, t in years.

If $P = 100,000$ when $t = 2$ and
 $P = 50,000$ when $t = 5$, then

what is the population of fish when $t = 10$?

$$\rightarrow \frac{dP}{dt} = kP \Rightarrow P(t) = ce^{kt}, \quad P(10) = ??$$

$$100,000 = P(2) = ce^{2k}$$

$$50,000 = P(5) = ce^{5k}$$

$$50,000 = ce^{5k}$$

$$100,000 = ce^{2k}$$

divide eq. 1 by eq. 2

$$\frac{1}{2} = \frac{50,000}{100,000} = \frac{ce^{5k}}{ce^{2k}} = e^{5k-2k} = e^{3k}$$

$$\frac{1}{2} = e^{3k}$$

$$\ln\left(\frac{1}{2}\right) = \ln(e^{3k})$$

$$\cancel{\ln 1} - \ln 2 = 3k$$

$$0 - \ln 2 = 3k$$

$$k = -\frac{1}{3} \ln 2 < 0 \quad \text{so exponential decay!}$$

$$50,000 = c e^{5(-\frac{1}{3} \ln 2)}$$

$$c = \frac{50,000}{e^{-\frac{5}{3} \ln 2}}$$

$$= \frac{50,000}{e^{\ln 2^{-5/3}}}$$

$$= \frac{50,000}{2^{-5/3}}$$

$$P(10) = c e^{k(10)} = \frac{50,000}{2^{-5/3}} e^{(-\frac{1}{3} \ln 2)(10)}$$

$\approx 15,749.01$ fish
after 10 years.

(2) The radio active isotope ^{226}Ra has a half life of about 1599 years.

There are 85g of ^{226}Ra now.

How much remains after 1400 years?

$$\text{half life} \Rightarrow P(t) = ce^{kt}$$

P = ammount of ^{226}Ra

t = time in years.

$$P(1400) = ??$$

$$85 = P(0) = ce^{k(0)} = ce^0 = c$$

$$P(t) = 85e^{kt}$$

$$\frac{85}{2} = P(1599) = 85e^{k(1599)}$$

$$\frac{1}{2} = e^{1599k}$$

$$\ln\left(\frac{1}{2}\right) = \ln(e^{1599k})$$

$$-\ln 2 = 1599k$$

$$k = \frac{-\ln 2}{1599}$$

$$\begin{aligned} P(1400) &= 85e^{\frac{-\ln 2}{1599} \cdot 1400} \end{aligned}$$

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Examples

① The radio active isotope ^{226}Ra has a half life of about 1599 years. There are 85g of ^{226}Ra now. How much remains after 1400 years?

half life $\Rightarrow P(t) = ce^{kt}$

P : amount of ^{226}Ra

t : time in years

$$P(1400) = ??$$

$$85 = P(0) = ce^{k(0)} = ce^0 = c$$

$$P(t) = 85e^{kt}$$

$$\frac{85}{2} = P(1599) = 85 e^{k(1599)}$$

$$\frac{1}{2} = e^{1599k}$$

$$\ln\left(\frac{1}{2}\right) = \ln(e^{1599k})$$

$$\cancel{\ln 1} - \ln 2 = 1599k$$

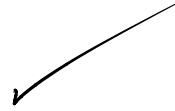
$$0 - \ln 2 = 1599k$$

$$k = \frac{-\ln 2}{1599} < 0$$

exponential
decay

$$P(1400) = 85 e^{\frac{-\ln 2}{1599} \cdot 1400}$$

$$\approx 46.33 \text{ g of } {}^{226}\text{Ra}$$



- ② The radioactive isotope ${}^{239}\text{Pu}$ has a half life of about 24100 years.
after 1500 years there are 5g of ${}^{239}\text{Pu}$.

a) What is the initial quantity of ${}^{239}\text{Pu}$?

What is $c = ??$

$$P(t) = ce^{kt} \quad \text{where}$$

P : amount of ${}^{239}\text{Pu}$
 t : time in years.

$$\frac{c}{2} = P(24100) = c e^{k(24100)}$$

$$\frac{1}{2} = e^{24100k}$$

$$\ln\left(\frac{1}{2}\right) = \ln(e^{24100k})$$

$$-\ln 2 = 24100k$$

$$k = \frac{-\ln 2}{24100} < 0$$

$$P(t) = c e^{\frac{-\ln 2}{24100} t}$$

$$5 = P(1500) = c e^{\frac{-\ln 2}{24100} \cdot 1500}$$

$$c = \frac{5}{e^{\frac{-1500}{24100} \cdot \ln 2}}$$

$$= 5 e^{\frac{15}{241} \ln 2}$$

$$= 5 e^{\ln 2^{15/241}}$$

$$= 5 \cdot 2^{15/241}$$

≈ 5.22 grams of ^{239}Pu initially