# Lecture 10: Chain rule

HW 9 #1(c) $f(x) = \dfrac{9\cot x}{4+2\cos x}$ ; find $f'(\frac{\pi}{2})$

$$f'(x) = \frac{d}{dx}\left[\frac{9\cot x}{4+2\cos x}\right] = \frac{(9\cot x)'(4+2\cos x) - 9\cot x\,(4+2\cos x)'}{(4+2\cos x)^2}$$

$$= \frac{-9\csc^2 x\,(4+2\cos x) - 9\cot x\,(-2\sin x)}{(4+2\cos x)^2}$$

$$f'(\tfrac{\pi}{2}) = \frac{-9(4+2\cdot 0) - 9\cdot 0\cdot(-2)}{(4+2\cdot 0)^2}$$

$$= -\frac{36}{16}$$

$$= -\frac{9}{4}$$

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Suppose $y = g(h(x)) = (g\circ h)(x)$.

Question: What is $dy/dx$ ?

we call $g$ outside fun.
call $h$ inside fun

**Chain Rule:**

$$\frac{dy}{dx} = \frac{dy}{dh}\cdot\frac{dh}{dx} \quad \text{or} \quad y'(x) = g'(h(x))\cdot h'(x)$$

**e.g.** ① $f(x) = 6(x^2+1)^{2023}$ ; find $f'(x)$

$g(u) = 6u^{2023}$
$h(x) = x^2+1$

$$g(h(x)) = 6\,h(x)^{2023}$$
$$= 6(x^2+1)^{2023} = f(x) \checkmark$$

$f'(x) = g'(h(x)) \cdot h'(x)$

$g'(u) = 6 \cdot 2023\, u^{2022}$
$h'(x) = 2x$

$f'(x) = 6 \cdot 2023 \left(x^2+1\right)^{2022} \cdot 2x$

$= 2 \cdot 6 \cdot 2023\, x\,(x^2+1)^{2022}$ ✓

② $f(x) = \sqrt[5]{x^3+3x-1}$ ; find $f'(x)$.

outside : $g(u) = \sqrt[5]{u} = u^{1/5}$
inside : $h(x) = x^3+3x-1$

$g(h(x)) = \sqrt[5]{h(x)} = \sqrt[5]{x^3+3x-1} = f(x)$ ✓

$f'(x) = g'(h(x)) \cdot h'(x)$

$g'(u) = \frac{1}{5} u^{-4/5}$ $\quad h'(x) = 3x^2+3$ ✓

$f'(x) = \frac{1}{5}\left(x^3+3x-1\right)^{-4/5} \cdot \left(3x^2+3\right)$

$$= \frac{3x^2+3}{5 \cdot \sqrt[5]{(x^3+3x-1)^4}}$$

③ $y = \left(\frac{8x^2}{x+1}\right)^3$ ; find $y'$

outside : $g(u) = u^3$

inside : $h(x) = \frac{8x^2}{x+1}$

$$y' = g'(h(x)) \cdot h'(x)$$

$g'(u) = 3u^2$

$$h'(x) = \frac{d}{dx}\left[\frac{8x^2}{x+1}\right]$$

$$= \frac{16x(x+1) - 8x^2 \cdot 1}{(x+1)^2}$$

$$= \frac{16x^2 + 16x - 8x^2}{(x+1)^2}$$

$$= \frac{8x^2 + 16x}{(x+1)^2}$$

$$y' = 3\left(\frac{8x^2}{x+1}\right)^2 \cdot \frac{8x^2+16x}{(x+1)^2}$$ ✓

④ $f(x) = (e^x + \sin x)^{2/3}$

outside : $g(u) = u^{2/3}$

inside : $h(x) = e^x + \sin x$

$$f'(x) = g'(h(x)) \cdot h'(x)$$

$g'(u) = \frac{2}{3} u^{-1/3}$ $\qquad$ $h'(x) = e^x + \cos x$

$$f'(x) = \frac{2}{3}\left(e^x + \sin x\right)^{-1/3} \cdot \left(e^x + \cos x\right)$$ ✓

$$= \frac{2(e^x + \cos x)}{3 \cdot \sqrt[3]{e^x + \sin x}}$$

5 $y = -(x^3 + 1 - e^x)^7$

outside: $g(u) = -u^7$

inside: $h(x) = x^3 + 1 - e^x$

$y' = g'(h(x)) \cdot h'(x)$

# Lecture 10: Chain rule

HW 9 # 71 $y = \frac{5(a^2 - x^2)}{a^2 + x^2}$ a is a constant. find $y'$.

a is just a number

$$y' = \frac{d}{dx}\left[\frac{5(a^2 - x^2)}{a^2 + x^2}\right] =$$

$$\frac{d}{dx}[5(a^2 - x^2)] = 5\frac{d}{dx}[a^2 - x^2]$$
$$= 5\left(\frac{d}{dx}[a^2] - \frac{d}{dx}[x^2]\right)$$
$$= 5(0 - 2x)$$
$$= -10x$$

$\frac{d}{dx}[3^2] = 0$

$$\frac{d}{dx}[a^2 + x^2] = \frac{d}{dx}[a^2] + \frac{d}{dx}[x^2] = 0 + 2x = 2x$$

$$y' = \frac{-10x(a^2 + x^2) - 5(a^2 - x^2)\cdot 2x}{(a^2 + x^2)^2}$$

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Suppose $y = g(h(x))$. What is the derivative of $y$?

**Chain rule:**

$$\frac{dy}{dx} = \frac{dy}{dh}\cdot\frac{dh}{dx} \quad \text{or} \quad y'(x) = g'(h(x))\cdot h'(x)$$

outside function : $g(u)$

inside function: $h(x)$

e.g. ① $f(x) = 6(x^2+1)^{2023}$; find $f'(x)$.

outside : $g(u) = 6u^{2023}$
inside : $h(x) = x^2+1$

$(g \circ h)(x) = g(h(x)) = 6(\ h(x)\ )^{2023}$
sanity check $= 6(\ x^2+1\ )^{2023} = f(x)$ ✓

$f'(x) = g'(h(x)) \cdot h'(x)$

$g'(u) = 6 \cdot 2023\, u^{2022}$ $\qquad h'(x) = 2x$

$f'(x) = 6 \cdot 2023\ (\ x^2+1\ )^{2022} \cdot 2x$ ✓

② $f(x) = \sqrt[5]{x^3+3x-1}$; find $f'(x)$.
outside : $g(u) = \sqrt[5]{u} = u^{1/5}$
inside : $h(x) = x^3+3x-1$

$g'(u) = \frac{1}{5}u^{-4/5}$ $\qquad h'(x) = 3x^2+3$

$f'(x) = g'(\ h(x)\ ) \cdot h'(x)$
$= \frac{1}{5}(x^3+3x-1)^{-4/5} \cdot (\ 3x^2+3)$ ✓

③ $y = \left( \dfrac{8x^2}{x+1} \right)^3$

outside : $g(u) = u^3$

inside : $h(x) = \dfrac{8x^2}{x+1}$

$g'(u) = 3u^2$

$$h'(x) = \frac{d}{dx}\left[\frac{8x^2}{x+1}\right]$$
$$= \frac{16x(x+1) - 8x^2(1)}{(x+1)^2}$$
$$= \frac{16x^2 + 16x - 8x^2}{(x+1)^2}$$
$$= \frac{8x^2 + 16x}{(x+1)^2}$$

$$\frac{d}{dx}[x+1]$$
$$= \frac{d}{dx}[x] + \frac{d}{dx}[1]$$
$$= 1 + 0 \checkmark$$

$$f'(x) = g'(h(x)) \cdot h'(x)$$
$$= 3\left(\frac{8x^2}{x+1}\right)^2 \cdot \frac{8x^2+16x}{(x+1)^2}$$

④ $f(x) = (e^x + \sin x)^{2/3}$ ; find $f'(x)$.

outside : $g(u) = u^{2/3}$

inside : $h(x) = e^x + \sin x$

$g'(u) = \frac{2}{3} u^{-1/3}$ $\qquad h'(x) = e^x + \cos x$

$$f'(x) = g'(h(x)) \cdot h'(x)$$
$$= \frac{2}{3}(e^x + \sin x)^{-1/3} \cdot (e^x + \cos x)$$

$\checkmark$