Lecture 11: The chain; Derivative of the natural log

The Chain Suppose $y = g(h(x)) = (g \circ h)(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dh} \cdot \frac{dh}{dx} \qquad y'(x) = g'(h(x)) \cdot h'(x)$$

e.g. ① $f(x) = (3x+1)^3 (x^4 - 3x)^{4/3}$; $f'(x) = ?$

$$f'(x) = \frac{d}{dx}\Big[\underbrace{(3x+1)^3}_{h(x)} \cdot \underbrace{(x^4-3x)^{4/3}}_{g(x)}\Big]$$

$$= h'(x)\,g(x) + h(x)\,g'(x)$$

$$= \frac{d}{dx}\left[(3x+1)^3\right](x^4-3x)^{4/3} + (3x+1)^3 \frac{d}{dx}\left[(x^4-3x)^{4/3}\right]$$

$$\frac{d}{dx}\left[(3x+1)^3\right] = f'(k(x)) \cdot k'(x)$$

outside $f(u) = u^3$ $\qquad f'(u) = 3u^2$

inside $k(x) = 3x+1$ $\qquad k'(x) = 3$

$$= 3(3x+1)^2 \cdot 3 = 9(3x+1)^2$$

$$\frac{d}{dx}\left[(x^4-3x)^{4/3}\right] = f'(k(x))\,k'(x)$$

outside $f(u) = u^{4/3}$ $\qquad f'(u) = \frac{4}{3}u^{1/3}$

inside $k(x) = x^4 - 3x$ $\qquad k'(x) = 4x^3 - 3$

$$= \frac{4}{3}(x^4-3x)^{1/3}(4x^3-3)$$

$$f'(x) = 9(3x+1)^2(x^4-3x)^{4/3} + (3x+1)^3 \cdot \frac{4}{3}(x^4-3x)^{1/3}(4x^3-3)$$

② $y = 3\csc(5x^3-2x^2+7)$; find y'

$$y' = \frac{d}{dx}\left[3\csc(5x^3-2x^2+7)\right] = g'(h(x))\,h'(x)$$

outside $g(u) = 3\csc u$ $\qquad g'(u) = -3\csc u \cot u$

inside $h(x) = 5x^3 - 2x^2 + 7$ $\qquad h'(x) = 15x^2 - 4x$

$$y' = -3\csc(5x^3-2x^2+7)\cot(5x^3-2x^2+7)\cdot(15x^2-4x)$$

$$-(45x^2-12x)\csc(5x^3-2x^2+7)\cot(5x^3-2x^2+7)$$

③ $y = 6\tan^2(4x)$; $y'(x) = ?$

$$y = 6(\tan(4x))^2$$

out $g(u) = 6u^2$

$g'(u) = 12u$

inside: $h(x) = \tan(4x)$

$= \tan(4x)$

$= \sec^2(4x)\cdot 4$

$= 4\sec^2(4x)$

$$y' = g'(h(x))\,h'(x)$$

$$= 12(\tan(4x))\cdot 4\sec^2(4x)$$

✓

Derivative of the natural log

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$$

e.g. 4 $y = \ln\sqrt{\frac{5x+1}{x^2-4}}$; find y'

$$y = \ln\left[\left(\frac{5x+1}{x^2-4}\right)^{1/2}\right]$$

$$= \frac{1}{2}\ln\left(\frac{5x+1}{x^2-4}\right)$$

$$= \frac{1}{2}\left[\ln(5x+1) - \ln(x^2-4)\right]$$

$$= \frac{1}{2}\ln(5x+1) - \frac{1}{2}\ln(x^2-4)$$

$$y' = \frac{d}{dx}\left[\frac{1}{2}\ln(5x+1)\right] - \frac{d}{dx}\left[\frac{1}{2}\ln(x^2-4)\right]$$

$$= \frac{1}{2}\frac{d}{dx}\left[\ln(5x+1)\right] - \frac{1}{2}\frac{d}{dx}\left[\ln(x^2-4)\right]$$

$$= \frac{1}{2}\frac{1}{5x+1}\cdot 5 \qquad - \frac{1}{2}\cdot\frac{1}{x^2-4}\cdot(2x)$$

$$= \frac{5}{10x+2} - \frac{x}{x^2-4}$$

$$= \frac{5(x^2-4) - x(10x+2)}{(10x+2)(x^2-4)} = \frac{-5x^2-2x-20}{(10x+2)(x^2-4)}$$ ✓

may need ?

Lecture 11: Chain Rule; Derivative of natural log

HW 10 #6 $y = \sqrt{3x^3 - 9x^2 - 9/x}$; y'?

$$y = (3x^3 - 9x^2 - 9x^{-1})^{1/2}$$

$$= \frac{1}{2}(3x^3 - 9x^2 - 9x^{-1})^{-1/2} \cdot (9x^2 - 18x + 9x^{-2})$$

$$= \frac{9x^2 - 18x + \frac{9}{x^2}}{2\sqrt{3x^3 - 9x^2 - \frac{9}{x}}} \quad \checkmark$$

Chain Rule Suppose $y = g(h(x)) = (g \circ h)(x)$

$$\frac{dy}{dx} = \frac{dy}{dh} \cdot \frac{dh}{dx} \quad \text{or} \quad y'(x) = g'(h(x))\, h'(x)$$

e.g. ① $f(x) = (3x+1)^3 (x^4 - 3x)^{4/3}$; find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[(3x+1)^3 (x^4 - 3x)^{4/3}\right]$$

$$= \frac{d}{dx}\left[(3x+1)^3\right](x^4 - 3x)^{4/3} + (3x+1)^3 \frac{d}{dx}\left[(x^4 - 3x)^{4/3}\right]$$

$$\frac{d}{dx}\left[(3x+1)^3\right] = g'(h(x))\, h'(x)$$

out $g(u) = u^3$ $\qquad g'(u) = 3u^2$

in $h(x) = 3x+1$ $\qquad h'(x) = 3$

$$= 3(3x+1)^2 \cdot 3 = 9(3x+1)^2$$

$$\frac{d}{dx}\left[(x^4-3x)^{4/3}\right] = g'(h(x))\cdot h'(x)$$

out $\quad g(u) = u^{4/3} \qquad g'(u) = \frac{4}{3}u^{1/3}$

in $\quad h(x) = x^4 - 3x \qquad h'(x) = 4x^3 - 3$

$$= \frac{4}{3}(x^4-3x)^{1/3}(4x^3-3)$$

$$f'(x) = 9(3x+1)^2(x^4-3x)^{4/3} + (3x+1)^3\cdot\frac{4}{3}(x^4-3x)^{1/3}(4x^3-3)$$

② $y = 3\csc(5x^3 - 2x^2 + 7)$; find y'

outside $\quad g(u) = 3\csc(u)$ $\quad$ inside $\quad h(x) = 5x^3 - 2x^2 + 7$

$g'(u) = -3\csc u \cot u \qquad h'(x) = 15x^2 - 4x$

$$y' = g'(h(x))\cdot h'(x)$$

$$= -3\csc(5x^3-2x^2+7)\cot(5x^3-2x^2+7)\cdot(15x^2-4x)$$

③ $y = 6\tan^2(4x)$; find y'.

$$y = 6(\tan(4x))^2$$

out $\quad g(u) = 6u^2$ $\quad$ inside $\quad h(x) = \tan 4x$

$g'(u) = 12u \qquad\qquad = \tan(4x)$

$h'(x) = \sec^2(4x)\cdot 4$

$= 4\sec^2(4x)$

$$y' = g'(h(x))\cdot h'(x) = 12(\tan 4x)\cdot 4\sec^2(4x)$$

Derivative of natural log

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$$

e.g. (4) $y = \ln\sqrt{\dfrac{5x+1}{x^2-4}}$; find $\dfrac{dy}{dx}$.

$$y = \ln\left[\left(\frac{5x+1}{x^2-4}\right)^{1/2}\right]$$

$\log(a^b) = b\log a$

$$= \frac{1}{2}\ln\left(\frac{5x+1}{x^2-4}\right)$$

$$= \frac{1}{2}\left[\ln(5x+1) - \ln(x^2-4)\right]$$

$$= \frac{1}{2}\ln(5x+1) - \frac{1}{2}\ln(x^2-4)$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left[\ln(5x+1)\right] - \frac{1}{2}\frac{d}{dx}\left[\ln(x^2-4)\right]$$

$$= \frac{1}{2}\frac{d}{dx}\left[\ln(5x+1)\right] - \frac{1}{2}\frac{d}{dx}\left[\ln(x^2-4)\right]$$

$$= \frac{1}{2}\,\frac{1}{5x+1}\cdot 5 - \frac{1}{2}\cdot\frac{1}{x^2-4}\cdot 2x$$

$$= \frac{5}{10x+2} - \frac{x}{x^2-4}$$

$$= \frac{5(x^2-4) - x(10x+2)}{(10x+2)(x^2-4)}$$

$$= \frac{-5x^2-2x-20}{(10x+2)(x^2-4)}$$

✓