Lecture 13 : Implicit differentiation

Explicit function : $y = f(x)$

- $y = 3\sin x$
- $g(x) = 10x + x^{3/4}$

Implicit function $f(x, y) = 0$

- $y - 3x = 0$
- $2x^4 = 4y^2 + 6x^2$
- $\sin(x + 3y) = 2xy$
- $\tan(x/y) = 10x$

Idea for differentiation of $f(x,y) = 0$ wrt. x.

treat y as an unknown function of x

$$\frac{d}{dx}[y] = \frac{dy}{dx} = y'$$

e.g. ① $y - 3x = 0$; find $\frac{dy}{dx}$

$$\frac{d}{dx}[y - 3x] = \frac{d}{dx}[0]$$

$$\frac{d}{dx}[y] - 3\frac{d}{dx}[x] = 0$$

$$\frac{dy}{dx} - 3 = 0$$

$$\frac{dy}{dx} = 3$$

$$y = 3x$$

$$\frac{dy}{dx} = \frac{d}{dx}[3x] = 3$$

② find the equation for the tangent line to $2x^4 = 4y^2 + 6x^2$ at $(2, \sqrt{2})$.

$y - y_1 = m(x - x_1)$

(x_1, y_1) = is a pt on the function

m = slope at (x_1, y_1)

$$(x_1, y_1) = (2, \sqrt{2})$$

$$m = \left.\frac{dy}{dx}\right|_{(2,\sqrt{2})}$$

$$\frac{d}{dx}\left[2x^4\right] = \frac{d}{dx}\left[4y^2 + 6x^2\right]$$

$$8x^3 = \frac{d}{dx}\left[4y^2\right] + 12x$$

out : $4u^2$

in : $g = y$

$$8x^3 = 8(y) \cdot \frac{dy}{dx} + 12x$$

$$8y \cdot \frac{dy}{dx} = 8x^3 - 12x$$

$$\frac{dy}{dx} = \frac{8x^3 - 12x}{8y}$$

$$= \frac{2x^3 - 3x}{2y}$$

$$m = \left.\frac{dy}{dx}\right|_{(2,\sqrt{2})} = \frac{2(2)^3 - 3(2)}{2(\sqrt{2})} = \frac{5}{\sqrt{2}}$$

$$y - \sqrt{2} = \frac{5}{\sqrt{2}}(x-2) \checkmark$$

③ $\sin(x+3y) = 2xy$; find $\frac{dy}{dx}$.

$$\frac{d}{dx}[\sin(x+3y)] = \frac{d}{dx}[2x \cdot y]$$

$$\cos(x+3y) \cdot \frac{d}{dx}[x+3y] = 2y + 2x\frac{dy}{dx}$$

$$\cos(x+3y) \cdot \left(1 + 3\frac{dy}{dx}\right) = 2y + 2x\frac{dy}{dx}$$

$$\cos(x+3y) + 3\frac{dy}{dx} \cdot \cos(x+3y) = 2y + 2x\frac{dy}{dx}$$

$$3 \cdot \frac{dy}{dx}\cos(x+3y) - 2x\frac{dy}{dx} = 2y - \cos(x+3y)$$

$$\frac{dy}{dx} \cdot \left(3\cos(x+3y) - 2x\right) = 2y - \cos(x+3y)$$

$$\frac{dy}{dx} = \frac{2y - \cos(x+3y)}{3\cos(x+3y) - 2x}$$

④ $\tan(x/y) = 10x$; find dy/dx

$$\frac{d}{dx}\left[\tan\left(\frac{x}{y}\right)\right] = \frac{d}{dx}[10x]$$

$$\sec^2\left(\frac{x}{y}\right) \cdot \frac{d}{dx}\left[\frac{x}{y}\right] = 10$$

$$\frac{d}{dx}\left[\frac{x}{y}\right] = \frac{1 \cdot y - x \cdot \frac{dy}{dx}}{y^2}$$

option 1: $\underbrace{xy^{-1}}$ ✓

option 2: quotient

$$\sec^2\left(\frac{x}{y}\right) \cdot \frac{y - x\frac{dy}{dx}}{y^2} = 10$$

$$y - x\frac{dy}{dx} = \frac{10y^2}{\sec^2(x/y)}$$

$$y - x\frac{dy}{dx} = 10y^2\cos^2(x/y)$$

$$-x\frac{dy}{dx} = 10y^2\cos^2(x/y) - y$$

$$\frac{dy}{dx} = \frac{y - 10y^2\cos^2(x/y)}{x}$$ ✓

5 $e^{xy} = 2x$; find y' $\left(= \frac{dy}{dx} \right)$

$$\frac{d}{dx}\left[e^{xy} \right] = \frac{d}{dx}\left[2x \right]$$

$$e^{xy} \cdot \frac{d}{dx}\left[x \cdot y \right] = 2$$

$$e^{xy}\left(1 \cdot y + x y' \right) = 2$$

$$y e^{xy} + x y' e^{xy} = 2$$

$$x y' e^{xy} = 2 - y e^{xy}$$

$$y' = \frac{2 - y e^{xy}}{x e^{xy}} \checkmark$$

Lecture 13 : Implicit differentiation

HW 12 #2 $\quad g(x) = 4e^{3x}\cos(4x)$; $\quad g''(x) = ?$

$$g'(x) = 12e^{3x}\cos(4x) + 4e^{3x}(-4\sin(4x))$$
$$= 12e^{3x}\cos(4x) - 16e^{3x}\sin(4x)$$

$$g''(x) = 36e^{3x}\cos(4x) + 12e^{3x}(-4\sin(4x))$$
$$- 16\cdot 3e^{3x}\sin(4x) - 16e^{3x}\cdot 4\cos(4x)$$

Explicit functions : $y = f(x)$

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- $g(x) = 10x + x^{3/4}$

Implicit functions : $f(x,y) = 0$

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$$m = \left.\frac{dy}{dx}\right|_{(2,\sqrt{2})}$$

$$\frac{d}{dx}\left[2x^4\right] = \frac{d}{dx}\left[4y^2 + 6x^2\right]$$

$$8x^3 = \frac{d}{dx}\left[4y^2\right] + 12x$$

out: $g(u) = 4u^2$

in: $h(x) = y$

$$8x^3 = 8(y)\cdot\frac{d}{dx}[y] + 12x$$

$$8x^3 = 8y\cdot\frac{dy}{dx} + 12x$$

$$8y\cdot\frac{dy}{dx} = 8x^3 - 12x$$

$$\frac{dy}{dx} = \frac{8x^3 - 12x}{8y}$$

$$= \frac{2x^3 - 3x}{2y}$$

$$m = \left.\frac{dy}{dx}\right|_{(2,\sqrt{2})} = \frac{2(2)^3 - 3(2)}{2(\sqrt{2})} = \frac{5}{\sqrt{2}}$$

$$y - \sqrt{2} = \frac{5}{\sqrt{2}}(x-2) \quad \checkmark$$

(3) $\sin(x+3y) = 2xy$; find dy/dx

$$\frac{d}{dx}[\sin(x+3y)] = \frac{d}{dx}[2x \cdot y]$$

$$\cos(x+3y) \cdot \frac{d}{dx}[x+3y] = 2y + 2x \cdot \frac{dy}{dx}$$

$$\cos(x+3y)\left(1 + 3\frac{d}{dx}[y]\right) =$$

$$\cos(x+3y) \cdot \left(1 + 3\frac{dy}{dx}\right) = 2y + 2x \cdot \frac{dy}{dx}$$

$$\cos(x+3y) + 3\cos(x+3y) \cdot \frac{dy}{dx} = 2y + 2x \cdot \frac{dy}{dx}$$

$$3\cos(x+3y) \cdot dy/dx - 2x \cdot dy/dx = 2y - \cos(x+3y)$$

$$(3\cos(x+3y) - 2x) \cdot dy/dx = 2y - \cos(x+3y)$$

$$\frac{dy}{dx} = \frac{2y - \cos(x+3y)}{3\cos(x+3y) - 2x}$$

(4) $\tan(x/y) = 10x$ find y' (same as $\frac{dy}{dx}$)

$$\frac{d}{dx}\left[\tan(x/y)\right] = \frac{d}{dx}\left[10x\right]$$

$$\sec^2\left(\frac{x}{y}\right)\cdot\frac{d}{dx}\left[\frac{x}{y}\right] = 10$$

$$\frac{d}{dx}\left[\frac{x}{y}\right] = \frac{1\cdot y - xy'}{y^2}$$

$$\sec^2\left(\frac{x}{y}\right)\cdot\frac{y - xy'}{y^2} = 10$$

$$y - xy' = \frac{10y^2}{\sec^2(x/y)}$$

$$y - xy' = 10y^2\cos^2(x/y)$$

$$-xy' = 10y^2\cos^2(x/y) - y$$

$$y' = \frac{y - 10y^2\cos^2(x/y)}{x}$$

y' is the same thing as $\frac{dy}{dx}$