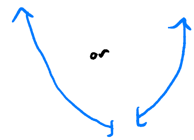
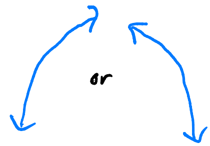
Lecture 18: Concavity, inflection points; Second derivative test.

Concavity

A function is **concave up** on interval (a, b) if $f''(x) > 0$ on (a, b)



A function is **concave down** on interval (a, b) if $f''(x) < 0$ on (a, b)



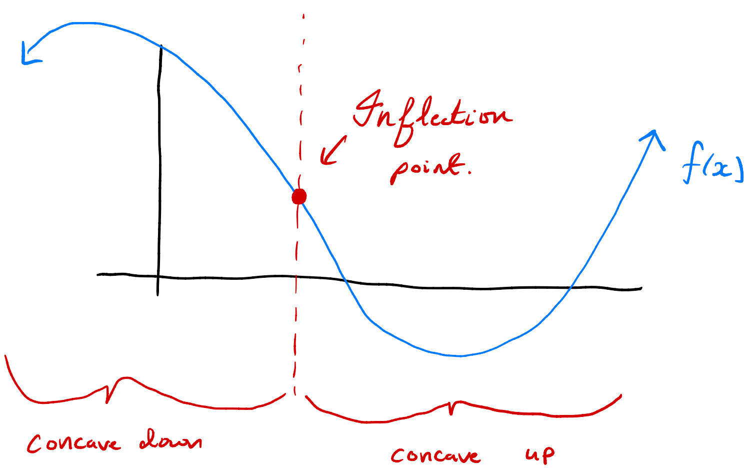
Inflection point

if $x = c$ has the property that f'' switches signs at $x = c$, then we say $(c, f(c))$ is an **inflection point** of f.

* note $x = c$ must be in the domain *

How to find inflection points?

1) find all points such that $f''(x) = 0$ or DNE
2) determine if f'' switches sign at these points.



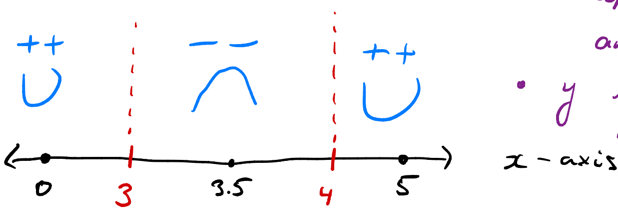
e.g. 1. $y = (x^2 - 11x + 32)e^x$ find all inflection pts.

$$y' = (2x - 11)e^x + (x^2 - 11x + 32)e^x$$
$$= e^x(x^2 - 9x + 21)$$

$$y'' = e^x(x^2 - 9x + 21) + e^x(2x - 9)$$
$$= e^x(x^2 - 7x + 12)$$

$y'' = 0$	y'' DNE
$0 = e^x(x^2 - 7x + 12)$	nothing here
$0 = x^2 - 7x + 12$	

$$0 = (x-4)(x-3)$$
$$x = 4 \text{ or } 3$$



- y is concave up on $(-\infty, 3)$ and $(4, \infty)$
- y is concave down on $(3, 4)$

$$y''(0) = e^0(0-4)(0-3) = 12$$

$$y''(3.5) = e^{3.5}(3.5-4)(3.5-3) < 0$$

$$y''(5) = e^5(5-4)(5-3) > 0$$

$(3, y(3))$ and $(4, y(4))$ are inflection pts.

$y(3) = (3^2 - 11 \cdot 3 + 32)e^3 = 8e^3$

$y(4) = (4^2 - 11 \cdot 4 + 32)e^4 = 4e^4$

$(3, 8e^3)$ and $(4, 4e^4)$ are the inflection pts.

Second derivative test

Suppose $x = c$ is a critical number for f

1) If $f''(c) > 0$, then $(c, f(c))$ is a rel. min.

2) If $f''(c) < 0$, then $(c, f(c))$ is a rel max.

(2) $y = 5 + 6x - 8x^3$. find the rel ext.

$y' = 6 - 24x^2$

$y' = 0$	y' DNE
$0 = 6 - 24x^2$ $24x^2 = 6$ $x^2 = 1/4$ $x = \pm 1/2$	Nothing here

Critical numbers: $x = -\frac{1}{2}, \frac{1}{2}$

$y'' = -48x$

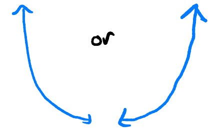
$y''(-\frac{1}{2}) = -48(-\frac{1}{2}) = 24 > 0$ rel min

$y''(\frac{1}{2}) = -48(\frac{1}{2}) = -24 < 0$ rel max.

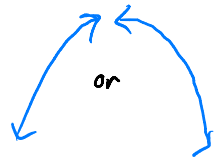
Lecture 18: Concavity and inflection points; Second derivative test.

Concavity

A function $f(x)$ is **concave up** on interval (a,b) if $f''(x) > 0$ on (a,b) $\underset{\cup}{++}$



A function $f(x)$ is **concave down** on an interval (a,b) if $f''(x) < 0$ on (a,b) $\underset{\cap}{--}$



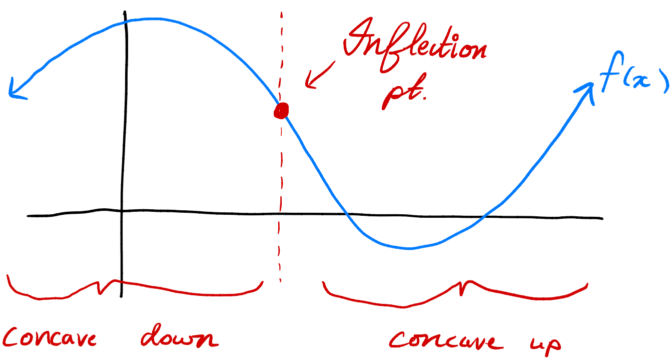
Inflection points

- If f is defined near c and f'' switches signs at $x=c$, then we say $(c, f(c))$ is an **inflection pt** of f.

How to find inflection pts?

1) find all numbers in the domain of f such that $f''=0$ or DNE

2) determine if concavity switches near these numbers.



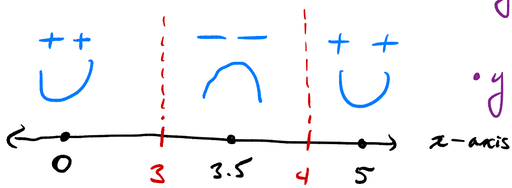
e.g. ① $y = (x^2 - 11x + 32)e^x$ find all inflection pts.

$$y' = (2x - 11)e^x + (x^2 - 11x + 32)e^x$$
$$= e^x(x^2 - 9x + 21)$$

$$y'' = e^x(x^2 - 9x + 21) + e^x(2x - 9)$$
$$= e^x(x^2 - 7x + 12)$$
$$= e^x(x - 3)(x - 4)$$

$y'' = 0$	y'' DNE
$0 = e^x(x-3)(x-4)$	nothing here
$0 = (x-3)(x-4)$	

$x = 3 \quad x = 4$



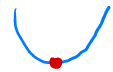
- y is concave up on $(-\infty, 3)$ and $(4, \infty)$
- y is concave down on $(3, 4)$

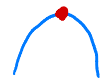
$$y''(0) = e^0(0 - 4)(0 - 3) > 0$$
$$y''(3.5) = e^{3.5}(3.5 - 4)(3.5 - 3) < 0$$
$$y''(5) = e^5(5 - 4)(5 - 3) > 0$$

So we have that $(3, y(3)) = (3, \overbrace{(3^2 - 11\cdot 3 + 32)e^3}^{8e^3})$
and $(4, y(4)) = (4, \underbrace{(4^2 - 11\cdot 4 + 32)e^4}_{4e^4})$
are inflection pts of f

Second derivative test

Suppose $x = c$ is a critical number of $f(x)$

1) If $f''(c) > 0$, then $(c, f(c))$ is rel min 

2) If $f''(c) < 0$, then $(c, f(c))$ is rel max 

② $y = 5 + 6x - 8x^3$. find all rel. ext.

$$y' = 6 - 24x^2$$

$y' = 0$	y' DNE
$0 = 6 - 24x^2$	nothing here.
$24x^2 = 6$	
$x^2 = 1/4$	
$x = \pm 1/2$	

Critical numbers : $x = -\frac{1}{2}$ and $x = \frac{1}{2}$

$$y'' = -48x$$

$y''(-\frac{1}{2}) = -48(-\frac{1}{2}) = 24 > 0$ rel min

$y''(\frac{1}{2}) = -48(\frac{1}{2}) = -24 < 0$ rel max.

$(-\frac{1}{2}, y(-\frac{1}{2}))$ is rel min pt.

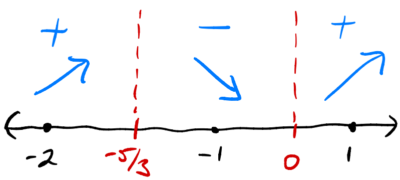
$(\frac{1}{2}, y(\frac{1}{2}))$ is rel max pt.

③ $f(x) = 2x^3 + 5x^2 + 2$ when is f concave up and increasing?

$f'(x) = 6x^2 + 10x$

$f'(x) = 0$	$f'(x)$ DNE
$0 = 6x^2 + 10x$	nothing here
$0 = 2x(3x+5)$	
$x = 0 \quad x = -\frac{5}{3}$	

Critical numbers : $x = -5/3 \quad x = 0$



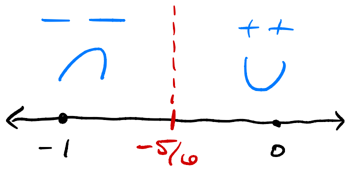
$f'(-2) = 6(-2)^2 + 10(-2) > 0$

$f'(-1) = 6(-1)^2 + 10(-1) < 0$

$f'(1) = 6(1)^2 + 10(1) > 0$

Now for concavity : $f''(x) = 12x + 10$

$f''(x) = 0$	f'' DNE
$0 = 12x + 10$	nothing here
$12x = -10$	
$x = -5/6$	



$$f''(-1) = 12(-1) + 10 < 0$$

$$f''(0) = 12(0) + 10 > 0$$

Thus f is **both** concave up and increasing on the interval $(0, \infty)$