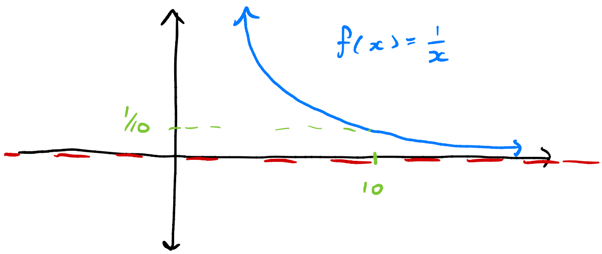
Lecture 21: Limits at infinity

$\lim_{x \to \infty} f(x)$: the value f approaches as $x \to \infty$

$\lim_{x \to -\infty} f(x)$: the value f approaches as $x \to -\infty$

e.g. ① $\lim_{x \to \infty} \frac{1}{x}$



x	10	100	1,000	10,000	$\xrightarrow{x}$ ∞
$f(x)$	0.1	0.01	0.001	0.0001	$\longrightarrow$ 0

$\lim_{x \to \infty} \frac{1}{x} = 0$ $\qquad$ $\lim_{x \to -\infty} \frac{1}{x} = 0$

② $\lim_{x \to \infty} \left(\frac{10x + 6}{11x^2 + 20} \right) \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x \to \infty} \frac{\frac{10}{x} + \frac{6}{x^2}}{11 + \frac{20}{x^2}}$

$= \frac{10 \lim_{x \to \infty} \left(\frac{1}{x}\right) + 6\left[\lim_{x \to \infty}\left(\frac{1}{x}\right)\right]^2}{\lim_{x \to \infty}(11) + 20 \lim_{x \to \infty}\left(\frac{1}{x^2}\right)}$ $\leftarrow$ $\lim_{x \to \infty} \frac{1}{x^2} = \left[\lim_{x \to \infty} \frac{1}{x}\right]^2$

$= \frac{10 \cdot 0 + 6 \cdot 0^2}{11 + 20 \cdot 0}$

$= \frac{0}{11} = 0$

(3) $\lim_{x \to -\infty} \left(\frac{3 - 7x^3}{x^3 + 2x^2 + 1} \right) \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$= \lim_{x \to -\infty} \frac{\frac{3}{x^3} - 7}{1 + \frac{2}{x} + \frac{1}{x^3}}$

$= \frac{\lim_{x \to -\infty} \frac{3}{x^3} - \lim_{x \to -\infty} 7}{\lim_{x \to -\infty} 1 + \lim_{x \to -\infty} \frac{2}{x} + \lim_{x \to -\infty} \frac{1}{x^3}}$

$\lim_{x \to -\infty} \frac{3}{x^3} = 3 \left[\lim_{x \to -\infty} \frac{1}{x} \right]^3 = 0$

$= \frac{0 - 7}{1 + 0 + 0} = \frac{-7}{1} = -7$

For Rational functions the limit as $x \to \infty$ ($x \to -\infty$) has 3 cases

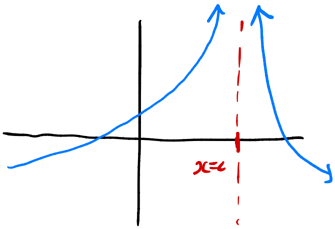
1) degree of top = degree of bottom
the limit is finit and the ratio of the leading coefficients

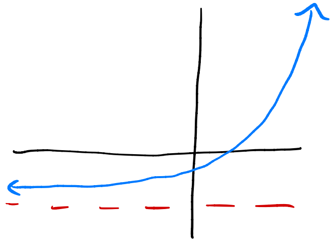
2) degree top > deg bottom, then
the limit is either $+\infty$ or $-\infty$

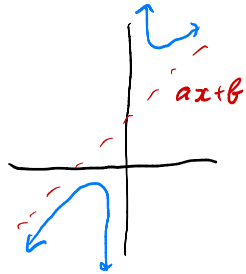
3) degree top < deg of bottom, then
the limit = 0

Asymptotes

$$y(x) = \frac{f(x)}{g(x)}$$







Verticle: $g(c) = 0$

horizontal

$$\lim_{x \to \infty} y = L$$

or

$$\lim_{x \to -\infty} y = L$$

slant

$\deg f > \deg g$

$$y(x) = ax + b + \frac{h(x)}{g(x)}$$

for some $h(x)$

④ find the asymptotes for

$$y = \frac{-4x^3 + 7x^2 + 22x + 28}{x^2 + 4}$$

vert: $x^2 + 4 = 0$ no sol.

so no vert. asym.

horz: none since $\lim_{x \to \infty} y$ and $\lim_{x \to -\infty} y$ are not finite.

slant:

$$
\begin{array}{r}
-4x + 7 \\
x^2 + 4 \overline{\big) -4x^3 + 7x^2 + 22x + 28} \\
-\left(-4x^3 \qquad - 16x\right) \\
\hline
0 + 7x^2 + 38x + 28 \\
-\left(7x^2 \qquad + 28\right) \\
\hline
0 + 38x + 0
\end{array}
$$

$38x$ = remainder #

$$\frac{-4x^3}{x^2} = -4x$$

$$\frac{7x^2}{x^2} = 7$$

$$\frac{-4x^3+7x^2+22x+28}{x^2+4} = -4x+7 + \frac{38x}{x^2+4}$$

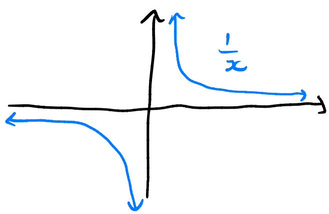
slant : $-4x+7$

Lecture 21: Limits at infinity

$\lim_{x\to\infty} f(x)$: the value f approaches as x becomes arbitrarily large ($x \to \infty$)

$\lim_{x\to-\infty} f(x)$: the value f approaches as x becomes arbitrarily negative ($x \to -\infty$).

e.g. ① $\lim_{x\to\infty} \frac{1}{x}$



x	1	10	100	1000	10,000	$\to \infty$
$\frac{1}{x}$	1	0.1	0.01	0.001	0.0001	$\to 0$

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

② $$\lim_{x\to\infty} \frac{10x+6}{11x^2+20} = \lim_{x\to\infty} \frac{10x}{11x^2} = \frac{10}{11} \lim_{x\to\infty} \frac{1}{x} = 0$$

③ $$\lim_{x\to-\infty} \frac{3-7x^3}{x^3+2x^2+1} = \lim_{x\to-\infty} \frac{-7x^3}{x^3} = \lim_{x\to-\infty} (-7) = -7$$

④ $$\lim_{x \to -\infty} \frac{7x^3 - 1}{20x^2 + x} = \lim_{x \to -\infty} \frac{7x^3}{20x^2}$$

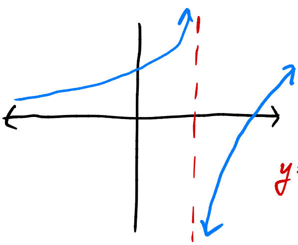
$$= \frac{7}{20} \lim_{x \to -\infty} x$$

$$= \frac{7}{20} (-\infty)$$

$$= -\infty$$

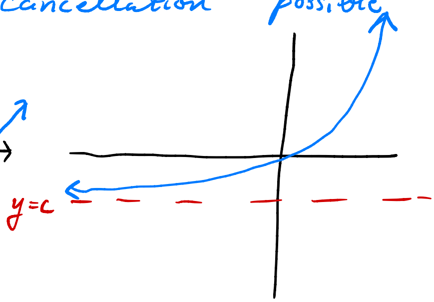
Asymptotes $\quad y(x) = \dfrac{f(x)}{g(x)}$

with no cancellation possible



Vert : $x = c$

if $g(c) = 0$



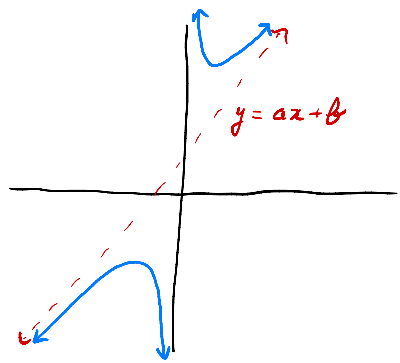
horz : $y = c$

$\lim_{x \to \infty} y = c$

or

$\lim_{x \to -\infty} y = c$

} not $\pm\infty$



slant : $y = ax + b$

deg f > deg g

$$\frac{f(x)}{g(x)} = ax + b + \frac{\text{remainder}}{g(x)}$$

⑤ find the asymtotes for $y = \frac{x^2+1}{x^2-25}$

Vert : $x^2 - 25 = 0$

$x^2 = 25$

$x = \pm 5$

vert. asy. $x = 5$ and $x = -5$

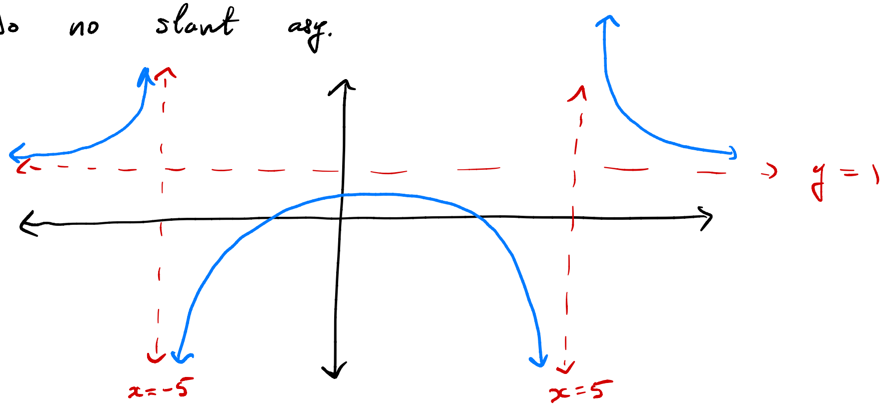
horz : $\lim_{x \to \infty} \frac{x^2+1}{x^2-25} = \lim_{x \to \infty} \frac{x^2}{x^2} = \lim_{x \to \infty} (1) = 1$

horz asy $y = 1$

slant : deg of $x^2+1 = 2$

deg of $x^2 - 25 = 2$

So no slant asy.



⑥ find the asy. for $y = \frac{-4x^3 + 7x^2 + 22x + 28}{x^2+4}$

$$\begin{array}{r} -4x + 7 \\ x^2 + 4 \overline{) -4x^3 + 7x^2 + 22x + 28} \\ -(-4x^3 - 16x) \\ \hline 0 + 7x^2 + 38x + 28 \\ -(7x^2 + 28) \\ \hline 0 + 38x + 0 \end{array}$$

remainder #

$\frac{-4x^3}{x^2} = -4x$

$\frac{7x^2}{x^2} = 7$

$\frac{38x}{x^2} = \frac{38}{x}$ ✗

$$\frac{-4x^3 + 7x^2 + 22x + 28}{x^2+4} = -4x + 7 + \frac{38x}{x^2+4}$$

slant

slant asy : $y = -4x + 7$.

Vert : $x^2 + 4 = 0$ no sol over real numbers.

So no vert. asy.

hor : $\lim_{x\to\infty} \frac{-4x^3 + 7x^2 + 22x + 28}{x^2+4} = \lim_{x\to\infty} \frac{-4x^3}{x^2}$

$= -4 \lim_{x\to\infty} x$

$= -4 \cdot \infty$

$= -\infty$

$\lim_{x\to-\infty} \frac{-4x^3 + 7x^2 + 22x + 28}{x^2+4} = \lim_{x\to-\infty} \frac{-4x^3}{x^2}$

$= -4 \lim_{x\to-\infty} x = -4(-\infty) = +\infty$

no hor. asy.