Lecture 26: Antiderivatives and indefinite integration

Antiderivates

$f(x) = 3x^2$. Is there a function $F(x)$ such that $F'(x) = 3x^2$?

Yes! $F(x) = x^3, \quad x^3 - 1, \quad x^3 + 100 - \pi, \ldots$

A function $F(x)$ is **an antiderivative** of $f(x)$ if $F'(x) = f(x)$.

* antiderivatives are not unique, they differ by a constant * (local constant)

Indefinite Integration

Let $F(x)$ be an antiderivative of $f(x)$, then the indefinite integral of $f(x)$ is

$$\underbrace{\int}_{\text{integral}} \underbrace{f(x)}_{\text{integrand}} \underbrace{dx}_{\text{which variable to integrate}} := F(x) + \underbrace{C}_{\text{constant of integration.}}$$

Basic Rules of Integration

- $\int 0 \, dx = c$
- $\int k \, dx = kx + c$
- $\boxed{\int k\, f(x)\, dx = k \int f(x)\, dx}$
- $\boxed{\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx}$

→ linearity of $\int$.

- $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c \qquad n \neq -1$
- $\int \cos x \, dx = \sin x + c$
- $\int \sin x \, dx = -\cos x + c$
- $\int \sec^2 x \, dx = \tan x + c$
- $\int \csc^2 x \, dx = -\cot x + c$
- $\int \sec x \tan x \, dx = \sec x + c$
- $\int \csc x \cot x \, dx = -\csc x + c$
- $\int e^x \, dx = e^x + c$
- $\int \frac{1}{x} \, dx = \ln|x| + c$

e.g. ① $\int 6x^2 + \frac{\sqrt[3]{x^2}}{6}\, dx$

$$= 6\int x^2\, dx + \frac{1}{6}\int x^{2/3}\, dx$$

$$= 6 \cdot \frac{1}{3} x^3 + \frac{1}{6} \cdot \frac{3}{5} x^{5/3} + C$$

$$= 2x^3 + \frac{1}{10} x^{5/3} + C$$

Check: $\left(2x^3 + \frac{1}{10} x^{5/3} + C\right)'$

$$= 6x^2 + \frac{1}{10} \cdot \frac{5}{3} x^{2/3} + 0$$

$$= 6x^2 + \frac{1}{6} \sqrt[3]{x^2}$$

② $\int 2 \tan x \cos x + 3\, dx$

$$= 2\int \tan x \cos x\, dx + 3\int dx$$

$$= 2\int \frac{\sin x}{\cos x} \cdot \cos x\, dx + 3\int dx$$

$$= 2\int \sin x\, dx + 3\int 1\, dx$$

$$= -2\cos x + 3x + C$$

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Yes! $F(x) = x^3,\ x^3+1,\ x^3 - 100 + \pi, \ldots$

A function $F(x)$ is **an antiderivative** of $f(x)$ if $F'(x) = f(x)$.

* antiderivatives are not unique, but they all differ by a (local) constant *

Indefinite Integral

Let $F(x)$ be an antiderivative of $f(x)$, then we define the **indefinite integral of $f(x)$** as

$$\underset{\text{integral}}{\int} \underset{\text{integrand}}{f(x)}\ \underset{\text{variable of integration}}{dx} := \underset{\text{antiderivative}}{F(x)} + \underset{\text{constant of integration}}{C}$$

Basic rules of integration

- $\int 0\,dx = c$
- $\int k\,dx = kx + c$
- $\int k\,f(x)\,dx = k\int f(x)\,dx$
- $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

the $\int$ is a linear operator.

- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ $\quad n \neq -1$
- $\int \cos x\,dx = \sin x + c$
- $\int \sin x\,dx = -\cos x + c$
- $\int \sec^2 x\,dx = \tan x + c$
- $\int \csc^2 x\,dx = -\cot x + c$
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- $\int e^x\,dx = e^x + c$
- $\int \frac{1}{x}\,dx = \ln|x| + c$

e.g. ① $\int 6x^2 + \frac{\sqrt[3]{x^2}}{6}\, dx$

$$= 6\int x^2\,dx + \frac{1}{6}\int x^{2/3}\,dx$$

$$= 6\,\frac{x^{2+1}}{2+1} + \frac{1}{6}\,\frac{x^{2/3+1}}{2/3+1} + C$$

$$= 2x^3 + \frac{1}{10}x^{5/3} + C$$

check: $\left(2x^3 + \frac{1}{10}x^{5/3} + C\right)' = 6x^2 + \frac{1}{10}\cdot\frac{5}{3}x^{2/3} + 0$

$$= 6x^2 + \frac{1}{6}\cdot\sqrt[3]{x^2}$$

② $\int 2\tan x \cos x\, dx$

$$= 2\int \frac{\sin x}{\cos x}\cdot \cos x\, dx = 2\int \sin x\, dx = -2\cos x + C$$

③ $\int \frac{1+7xe^x}{x} + 2\, dx$

$$= \int \frac{1}{x} + \frac{7xe^x}{x} + 2\, dx$$

$$= \int \frac{1}{x}dx + 7\int e^x dx + 2\int 1\,dx$$

$$= \ln|x| + 7e^x + 2x + C$$

(4) $\int \sec x(\sec x + 3\tan x)\,dx$

$= \int \sec^2 x + 3\sec x \tan x \, dx$

$= \int \sec^2 x \, dx + 3\int \sec x \tan x \, dx$

$= \tan x + 3\sec x + C$