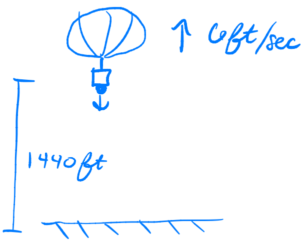
Lecture 28: Area and Riemann sums

HW 27 # 8



Find the time it takes the ball to hit the ground.

$a(t) = -32$

$v(t) = \int -32\, dt = -32t + c_1$

$6 = v(0) = -32(0) + c_1$

$c_1 = 6$

$s(t) = \int v(t)\, dt = \int -32t + 6\, dt = -16t^2 + 6t + c_2$

$1440 = s(0) = -16(0)^2 + 6(0) + c_2$

$c_2 = 1440$

$s(t) = -16t^2 + 6t + 1440$

$0 = -16t^2 + 6t + 1440$

$t \approx 9.68$ seconds

Sigma Notation

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2$$

ending value →

index →

$$\sum_{i=1}^{9} i^2 = \underbrace{1^2}_{i=1} + \underbrace{2^2}_{i=2} + \underbrace{3^2}_{i=3} + \underbrace{4^2}_{i=4} + \underbrace{5^2}_{i=5} + \underbrace{6^2}_{i=6} + \underbrace{7^2}_{i=7} + \underbrace{8^2}_{i=8} + \underbrace{9^2}_{i=9}$$

starting value ↑ ↑ rule

e.g. ① Expand $\displaystyle\sum_{i=3}^{6} i(\sqrt{i}+4)$

$$\sum_{i=3}^{6} i(\sqrt{i}+4) = \underbrace{3(\sqrt{3}+4)}_{i=3} + \underbrace{4(\sqrt{4}+4)}_{i=4} + \underbrace{5(\sqrt{5}+4)}_{i=5} + \underbrace{6(\sqrt{6}+4)}_{i=6}$$

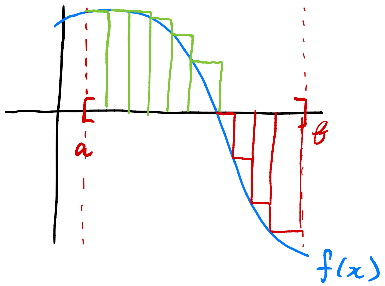
② Write

$$\underbrace{(1-1)^3}_{i=1} + \underbrace{(2-1)^3}_{i=2} + (3-1)^3 + (4-1)^3 + \cdots + \underbrace{(n-1)^3}_{i=n}$$

in sigma notation.

$$\sum_{i=1}^{n} (i-1)^3$$

Riemann sums



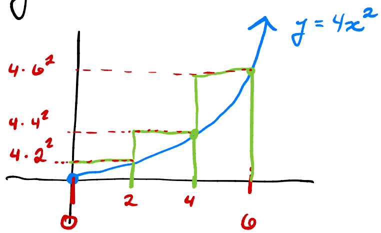
We have a curve $f(x)$ to approximate its signed area using rectangles on $[a, b]$.

sign area : area above x-axis is positive
area below x-axis is negative

Signed area of $f(x)$ = $\sum$ area of the rectangles above
− $\sum$ area of the rectangles below

e.g. 3 Approximate the signed area under $y = 4x^2$ on $[0, 6]$ use 3 rectangles.

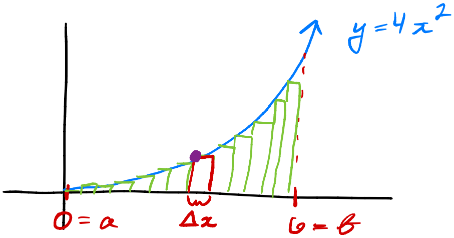
Right Riemann sums:



$$(4 \cdot 2^2)(2) + (4 \cdot 4^2)(2) + (4 \cdot 6^2)(2)$$

Left Riemann sum:

Now use 80 rectangles



$$\Delta x = \frac{b-a}{80} = \frac{6-0}{80} = \frac{6}{80} \leftarrow \text{width}$$

$$\sum_{i=0}^{79} \underbrace{4(0+i\Delta x)^2}_{\text{height}} \underbrace{\Delta x}_{\text{width}} = \sum_{i=0}^{79} 4\left(i \cdot \frac{6}{80}\right)^2 \frac{6}{80}$$

General Case: $f(x)$ on $[a, b]$ n rectangles

Right Riemann Sum

$$\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

Left Riemann Sum

$$\sum_{i=0}^{n-1} f(a+i\Delta x)\Delta x$$

Lecture 28: Area and Riemann sums

HW 27 # 51 $y'' = 3e^x + 4$ with $y'(0) = 3$ and $y(2) = 3e^2$
Find $y(3)$.

$$y' = \int y''\,dx = \int 3e^x + 4\,dx = 3e^x + 4x + c_1$$

$$3 = y'(0) = 3e^0 + 4(0) + c_1$$

$$3 = 3 + c_1$$
$$c_1 = 0$$

$$y = \int y'\,dx = \int 3e^x + 4x + 0\,dx = 3e^x + 2x^2 + c_2$$

$$3e^2 = y(2) = 3e^2 + 2(2)^2 + c_2$$

$$0 = 8 + c_2$$
$$c_2 = -8$$

$$y = 3e^x + 2x^2 - 8$$

$$y(3) = 3e^3 + 2(3)^2 - 8 = 3e^3 + 10$$

Sigma Notation

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$$

ending value →

$$\sum_{i=1}^{10} i^2 = \underset{i=1}{1^2} + \underset{i=2}{2^2} + \underset{i=3}{3^2} + \underset{\cdots}{\cdots} + \underset{i=10}{10^2}$$

index → $i=1$ ↑ rule

starting value

e.g.

1. Expand $\sum_{i=3}^{6} i(\sqrt{i}+4)$

$$= \underset{i=3}{3(\sqrt{3}+4)} + \underset{i=4}{4(\sqrt{4}+4)} + \underset{i=5}{5(\sqrt{5}+4)} + \underset{i=6}{6(\sqrt{6}+4)}$$

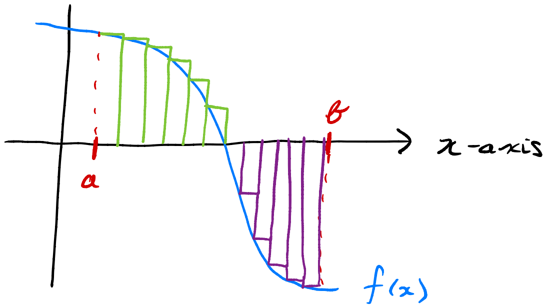
2. $$\underset{i=1}{(1-1)^3} + \underset{i=2}{(2-1)^3} + (3-1)^3 + \cdots + (n-1)^3$$

$$= \sum_{i=1}^{n} (i-1)^3$$

Riemann Sums

We have a function $f(x)$ and we want to compute the signed area of $f(x)$ on the interval $[a, b]$.

net change



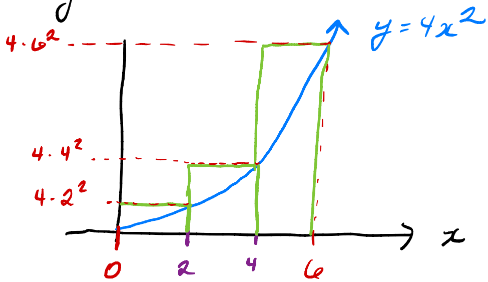
Signed area convention:

- area above x-axis is $+$
- area below x-axis is $-$

Signed area $\approx$ $\sum$ area of rect. above x-axis $- \sum$ area of rect. below x-axis

e.g. 3 Approx. signed area of $y = 4x^2$ on $[0, 6]$ using 3 rectangles.

Right Riemann Sum



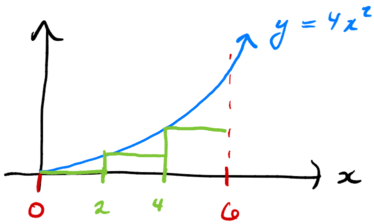
width of rect

$$\Delta x = \frac{6-0}{3} = 2$$

$$(4 \cdot 2^2)(2) + (4 \cdot 4^2)(2) + (4 \cdot 6^2)(2)$$

$$= \sum_{i=1}^{3} 4(i\Delta x)^2 \Delta x$$

Left Riemann Sum



$$(0)(2) + (4\cdot 2^2)(2) + (4\cdot 4^2)(2)$$

$$= \sum_{i=0}^{2} 4(i\Delta x)^2 \Delta x$$

General Case : $f(x)$ on $[a, b]$ use n rectangles

Right: $\displaystyle\sum_{i=1}^{n} f(a + i\Delta x)\Delta x$

Left : $\displaystyle\sum_{i=0}^{n-1} f(a + i\Delta x)\Delta x$

$$\Delta x = \frac{b-a}{n}$$