Lecture 3: Finding limits Analytically

$$\lim_{x \to c} f(x) = ?$$

Case 1: $f(x)$ $\overbrace{\text{does not have breaks/holes}}^{\text{continuous}}$, $f(c)$ is a number

$$\lim_{x \to c} f(x) = f(c)$$

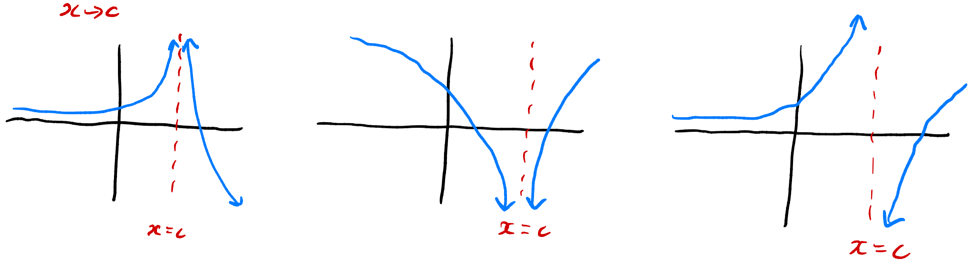
e.g. ① $f(x) = 3x^2 - 1$; $\lim_{x \to 2} f(x) = f(2) = 3 \cdot 2^2 - 1 = 11$

② $f(x) = \cos x$; $\lim_{x \to \pi/2} f(x) = \cos\left(\frac{\pi}{2}\right) = 0$

Case 2: $f(c) = \dfrac{\text{nonzero number}}{0}$

f has a verticle asymptote at $x = c$

$\lim_{x \to c} f(x)$ is either $+\infty$, $-\infty$, DNE



$+\infty$ $\qquad$ $-\infty$ $\qquad$ DNE

e.g. ③ $f(x) = \dfrac{1}{(x-5)^2}$; $\lim_{x \to 5} f(x)$

$$f(5) = \frac{1}{(5-5)^2} = \frac{1}{0}$$

x	4.9	4.99	5	5.01	5.1
$f(x)$	100	10,000		10,000	100

$$\lim_{x \to 5^-} f = +\infty = \lim_{x \to 5^+} f$$

$$\lim_{x \to 5} f(x) = +\infty$$

Case 3: $f(c) = \frac{0}{0}$

Either "hole at c" or "asymptote at c"

eg. (4) $f(x) = \dfrac{x^2 + 5x + 6}{x^2 + 3x + 2}$; $\lim_{x \to -2} f(x)$

$$f(-2) = \frac{4 - 10 + 6}{4 - 6 + 2} = \frac{0}{0}$$

$$f(x) = \frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+1)} \overset{*}{=} \frac{x+3}{x+1}$$

*only when $x \neq -2$

$$\frac{(-2) + 3}{(-2) + 1} = \frac{1}{-1} = -1$$

$$\lim_{x \to -2} \frac{(x+2)(x+3)}{(x+2)(x+1)} = \lim_{x \to -2} \frac{x+3}{x+1} = -1$$

Limit Properties

let $\lim_{x \to c} f(x) = L$ $\qquad \lim_{x \to c} g(x) = K$

- $\lim_{x \to c} \left(a f(x) \right) = a L$ $\qquad$ a number

- $\lim_{x \to c} \left(f(x) + g(x) \right) = L + K$

- $\lim_{x \to c} \left(f(x) \cdot g(x) \right) = L \cdot K$

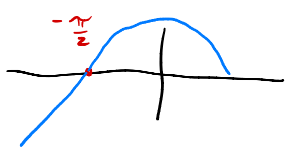
- if $K \neq 0$, $\lim_{x \to c} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{L}{K}$

- $\lim_{x \to c} \left([f(x)]^n \right) = L^n$

e.g. 5 $\quad f(x) = \begin{cases} x + \pi/2 & x \leq -\pi/2 \\ \cos x & -\pi/2 < x < \pi/2 \\ 4 \sin x + \pi & x \geq \pi/2 \end{cases}$

$\lim_{x \to -\frac{\pi}{2}} f(x) = ?$ $\qquad \lim_{x \to \frac{\pi}{2}} f(x) = ?$

$x \to -\frac{\pi}{2}$



$$\lim_{x \to -\frac{\pi}{2}^-} f(x) = \lim_{x \to -\frac{\pi}{2}^-} \left(x + \frac{\pi}{2} \right) = 0$$

$$\lim_{x \to -\frac{\pi}{2}^+} f(x) = \lim_{x \to -\frac{\pi}{2}^+} \cos x = 0$$

$\lim_{x \to -\frac{\pi}{2}} f(x) = 0$

Lecture 3: Finding limits analytically

HW2 #7 $f(x) = \dfrac{6e^x - 6}{x}$; $\lim_{x\to 0} f(x)$

x	-0.1	-0.01	0	0.01	0.1
$f(x)$	5.7098	5.9701	$-$	6.0301	6.3103

$\longrightarrow 6 \qquad 6 \longleftarrow$

$$\lim_{x\to 0} f(x) = 6$$

$$\lim_{x\to c} f(x) = ?$$

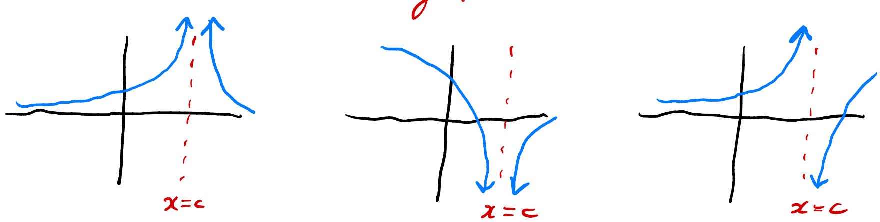
Case 1 f has no holes/breaks and $f(c)$ is a number.

e.g. ① $f(x) = \dfrac{1}{x^2+1}$; $\lim_{x\to -1} f(x)$

$$f(-1) = \frac{1}{(-1)^2+1} = \frac{1}{2} \qquad \lim_{x\to -1} f(x) = f(-1) = \frac{1}{2}$$

Case 2 $f(c) = \dfrac{\text{nonzero number}}{0}$

f has a vertical asymptote at $x = c$



$\lim = +\infty \qquad \lim = -\infty \qquad \lim$ DNE e.g. ② $f(x) = \dfrac{1}{(x-5)^2}$; $\lim\limits_{x \to 5} f(x)$

$$f(5) = \frac{1}{(5-5)^2} = \frac{1}{0}$$

x	4.9	4.99	5	5.01	5.1
$f(x)$	100	10,000	–	10,000	100

$\rightarrow +\infty$ $\quad +\infty \quad$ $\leftarrow$

$$\lim_{x \to 5} f(x) = +\infty$$

Case 3 $\quad f(c) = \dfrac{0}{0}$

Either f has a "hole at c" or an "asymptote at c"

e.g. ③ $f(x) = \dfrac{x^2 + 5x + 6}{x^2 + 3x + 2}$ $\qquad \lim\limits_{x \to -2} f(x)$

$$f(-2) = \frac{4 - 10 + 6}{4 - 6 + 2} = \frac{0}{0}$$

$$f(x) = \frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+1)} \overset{*}{=} \frac{x+3}{x+1} \qquad * x \neq -2$$

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x+3}{x+1} = \frac{-2+3}{-2+1} = -1$$

Limit Properties

$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} g(x) = K$

- $\lim_{x \to c} (a f(x)) = aL \qquad a$ number
- $\lim_{x \to c} (f(x) + g(x)) = L + K$
- $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot K$
- $K \neq 0$, then $\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{K}$
- $\lim_{x \to c} ([f(x)]^n) = L^n$

eg. 4 $f(x) = \begin{cases} x + \frac{\pi}{2} & x \leq -\pi/2 \\ \cos x & -\pi/2 < x < \pi/2 \\ 4\sin x + \pi & x \geq \pi/2 \end{cases}$

$\lim_{x \to -\frac{\pi}{2}} f(x) = ?$

$\lim_{x \to -\frac{\pi}{2}^-} f(x) = \lim_{x \to -\frac{\pi}{2}^-} \left(x + \frac{\pi}{2}\right) = -\frac{\pi}{2} + \frac{\pi}{2} = 0$

$\lim_{x \to -\frac{\pi}{2}^+} f(x) = \lim_{x \to -\frac{\pi}{2}^+} \cos x = \cos\left(-\frac{\pi}{2}\right) = 0$

$\lim_{x \to -\frac{\pi}{2}} f(x) = 0$

$$\lim_{x \to \frac{\pi}{2}} f(x) = ? \qquad f(x) = \begin{cases} x + \frac{\pi}{2} & x \leq -\pi/2 \\ \cos x & -\pi/2 < x < \pi/2 \\ 4\sin x + \pi & x \geq \pi/2 \end{cases}$$

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \cos x = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} (4\sin x + \pi) = 4\sin\left(\frac{\pi}{2}\right) + \pi = 4 + \pi$$

$$\lim_{x \to \frac{\pi}{2}} f(x) \quad \text{DNE}$$