Lecture 31: The fundamental theorem of calculus I

Recall: Let $f(x)$ be a function with an antiderivative $F(x)$ ($F'(x) = f(x)$), then

Indefinite integral : $\int f(x)\,dx := F(x) + c$

Definite integral : $\int_a^b f(x)\,dx :=$ signed area between $f(x)$ and x-axis on $[a, b]$.

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a) =: F(x)\Big|_a^b$$

e.g. ① Evaluate $\int_1^2 36x^3 + 9\,dx$

$$\int_1^2 36x^3 + 9\,dx = \left(\frac{36}{4}x^4 + 9x\right)\Big|_1^2$$

$$= \frac{36}{4}(2)^4 + 9(2) - \left[\frac{36}{4}(1)^4 + 9(1)\right]$$

$$= 144$$

② Evaluate $\int_0^6 3e^x + 9\,dx$

$$\int_0^6 3e^x + 9\,dx = \left(3e^x + 9x\right)\Big|_0^6$$

$$= 3e^6 + 9(6) - \left[3e^0 + 9(0)\right]$$

$$= 3e^6 + 51$$

③ Evaluate $\int_1^4 \frac{x^2+x^3}{\sqrt{x}}\,dx$

$$\int_1^4 \frac{x^2+x^3}{x^{1/2}}\,dx = \int_1^4 \frac{x^2}{x^{1/2}} + \frac{x^3}{x^{1/2}}\,dx$$

$$= \int_1^4 x^{3/2} + x^{5/2}\,dx$$

$$= \left(\frac{2}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right)\Big|_1^4$$

$$= \frac{2}{5}(4)^{5/2} + \frac{2}{7}(4)^{7/2} - \left[\frac{2}{5}(1)^{5/2} + \frac{2}{7}(1)^{7/2}\right]$$

$$= \frac{2}{5}\cdot 2^5 + \frac{2}{7}\cdot 2^7 - \left[\frac{2}{5} + \frac{2}{7}\right]$$

$$= \frac{2}{5}\cdot 2^5 + \frac{2}{7}\cdot 2^7 - \frac{2}{5} - \frac{2}{7}$$

$$= \frac{2}{5}(2^5 - 1) + \frac{2}{7}(2^7 - 1)$$

$$= 1704/35$$

4) Evaluate $\int_{-\pi/3}^{\pi/3} 9\tan x \cos x + 9\, dx$

$$\int_{-\pi/3}^{\pi/3} 9\tan x \cos x + 9\, dx = \int_{-\pi/3}^{\pi/3} 9\, \frac{\sin x}{\cos x} \cdot \cos x + 9\, dx$$

$$= \int_{-\pi/3}^{\pi/3} 9\sin x + 9\, dx$$

$$= \left(-9\cos x + 9x \right) \Big|_{-\pi/3}^{\pi/3}$$

$$= -9\cos\left(\frac{\pi}{3}\right) + 9\left(\frac{\pi}{3}\right) - \left[-9\cos\left(-\frac{\pi}{3}\right) + 9\left(-\frac{\pi}{3}\right) \right]$$

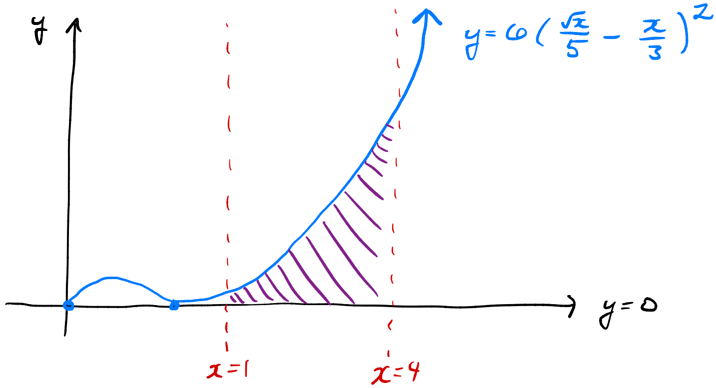
$$= -9\left(\tfrac{1}{2}\right) + 3\pi - \left[-9\left(\tfrac{1}{2}\right) - 3\pi \right]$$

$$= -9\left(\tfrac{1}{2}\right) + 3\pi + 9\left(\tfrac{1}{2}\right) + 3\pi$$

$$= 6\pi$$

(5) Find the area enclosed by the following graphs:

$$y = 6\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2 ; \quad y = 0 ; \quad x = 1 ; \quad x = 4$$



$$\int_1^4 6\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2 dx = \int_1^4 6\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right) dx$$

$$= 6\int_1^4 \frac{x}{25} - \frac{2x^{3/2}}{15} + \frac{x^2}{9}\, dx$$

$$= 6\left(\frac{1}{50}x^2 - \frac{2}{15}\cdot\frac{2}{5}x^{5/2} + \frac{1}{27}x^3\right)\Big|_1^4$$

$$= 6\left[\frac{1}{50}(4)^2 - \frac{4}{75}(4)^{5/2} + \frac{1}{27}(4)^3 - \left[\frac{1}{50}(1)^2 - \frac{4}{75}(1)^{5/2} + \frac{1}{27}(1)^3\right]\right]$$

$$= 5.88$$

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$$= \left(\frac{2}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right)\Big|_1^4$$

$$= \frac{2}{5}(4)^{5/2} + \frac{2}{7}(4)^{7/2} - \left[\frac{2}{5}(1)^{5/2} + \frac{2}{7}(1)^{7/2}\right]$$

$$= \frac{2}{5}\cdot 2^5 + \frac{2}{7}\cdot 2^7 - \frac{2}{5} - \frac{2}{7}$$

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$$\int_{-\pi/3}^{\pi/3} 9\tan x\cos x + 9\,dx$$

$$= \int_{-\pi/3}^{\pi/3} 9\,\frac{\sin x}{\cos x}\cdot\cos x + 9\,dx$$

$$= \int_{-\pi/3}^{\pi/3} 9\sin x + 9\,dx$$

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$$= -9\cos\left(\frac{\pi}{3}\right) + 9\left(\frac{\pi}{3}\right) - \left[-9\cos\left(-\frac{\pi}{3}\right) + 9\left(-\frac{\pi}{3}\right)\right]$$

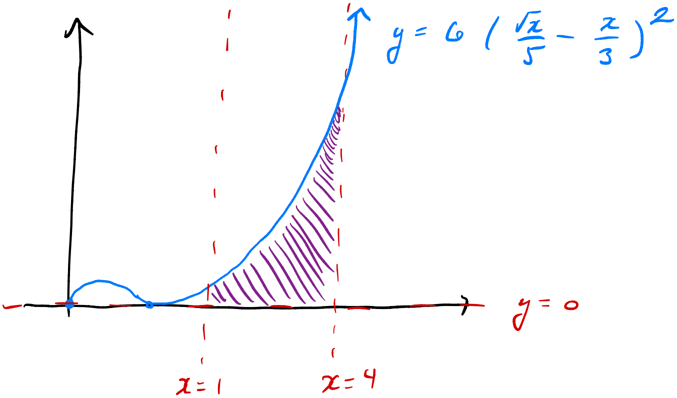
$$= -9\left(1/2\right) + 3\pi - \left[-9\left(1/2\right) - 3\pi\right]$$

$$= -9(1/2) + 3\pi + 9(1/2) + 3\pi$$

$$= 6\pi$$

⑤ Find the area by the graphs:

$$y = 6\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2 ; \quad y = 0 ; \quad x = 1 ; \quad x = 4$$



$$\text{area } (= \text{signe area}) = \int_1^4 6\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)^2 dx$$

$$= 6 \int_1^4 \left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right)\left(\frac{\sqrt{x}}{5} - \frac{x}{3}\right) dx$$

$$= 6 \int_1^4 \frac{x}{25} - \frac{2}{15} x^{3/2} + \frac{x^2}{9} \, dx$$

$$= 6 \left(\frac{1}{50} x^2 - \frac{2}{15} \cdot \frac{2}{5} x^{5/2} + \frac{1}{27} x^3 \right) \Big|_1^4$$

$$= 6 \left[\frac{1}{50}(4)^2 - \frac{4}{75}(4)^{5/2} + \frac{1}{27}(4)^3 - \left[\frac{1}{50}(1)^2 - \frac{4}{75}(1)^{5/2} + \frac{1}{27}(1)^3 \right] \right]$$

$$= 6 \left[\frac{16}{50} - \frac{2^7}{75} + \frac{4^3}{27} - \frac{1}{50} + \frac{4}{75} - \frac{1}{27} \right]$$

$$= 6 \left[\frac{15}{50} - \frac{124}{75} + \frac{63}{27} \right]$$

$$= 5.88$$