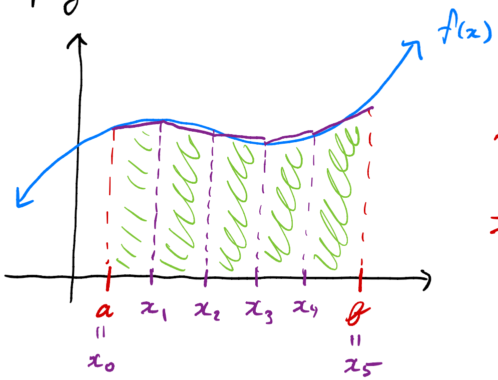
Lecture 33: Numerical integration

Recall: We computed $\int_a^b f(x)\,dx$ using Fundamental Theorem of Calculus (FTC), but we needed an antiderivative of $f(x)$.

Trapezoid rule



n = # of trapezoids

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$$\int_a^b f(x)\,dx \approx$$

$$\underbrace{\frac{f(x_0)+f(x_1)}{2}\Delta x}_{\text{area of 1st trapezoid}} + \frac{f(x_1)+f(x_2)}{2}\Delta x + \cdots + \frac{f(x_4)+f(x_5)}{2}\Delta x$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5) \right]$$

For general n, $\quad \Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$

$$\int_a^b f(x)\,dx \approx T_n := \frac{\Delta x}{2}\Big[f(x_0) + \underbrace{2f(x_1) + \cdots + 2f(x_{n-1})}_{\text{multiplied by 2}} + f(x_n) \Big]$$

e.g. ① Approximate $\int_3^5 7x^2+1\,dx$ using the trapezoid rule with $n=4$.

$f(x) = 7x^2+1$ $\qquad \Delta x = \frac{5-3}{4} = \frac{1}{2}$ $\qquad x_i = a + i\Delta x$

$\qquad\qquad = 3 + i(1/2)$

$$\int_3^5 7x^2+1\,dx \approx T_4$$

$$= \frac{1/2}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{4}\left[7(3)^2+1 + 2\left(7\left(\tfrac{7}{2}\right)^2+1\right) + 2\left(7(4)^2+1\right) + 2\left(7\left(\tfrac{9}{2}\right)^2+1\right) + 7(5)^2+1\right]$$

② Approximate $\int_{-2}^{2} e^{x^2}\,dx$ using the trapezoid rule with $n=4$.

$f(x) = e^{x^2}$ $\qquad \Delta x = \frac{2-(-2)}{4} = \frac{4}{4} = 1$

$$x_i = a + i\Delta x$$
$$= -2 + i$$

$x_0 = -2 + 0 = -2$

$x_1 = -2 + 1 = -1$ $\qquad x_3 = -2 + 3 = 1$

$x_2 = -2 + 2 = 0$ $\qquad x_4 = -2 + 4 = 2$

$$\int_{-2}^{2} e^{x^2}\,dx \approx T_4$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{2}\left[e^{(-2)^2} + 2e^{(-1)^2} + 2e^{0^2} + 2e^{1^2} + e^{2^2}\right]$$

$$= \frac{1}{2}\left[e^4 + 2e + 2 + 2e + e^4\right]$$

$$= \frac{1}{2}\left[2e^4 + 4e + 2\right] = e^4 + 2e + 1$$

(3) Approximate $\int_4^5 \sqrt{x^2+3}\,dx$ using trapezoid rule and $n = 3$.

$$f(x) = \sqrt{x^2+3} \qquad \Delta x = \frac{5-4}{3} = \frac{1}{3}$$

$$x_i = a + i\Delta x$$
$$= 4 + i(1/3)$$

$$x_0 = 4 + 0(1/3) = 4$$
$$x_1 = 4 + 1(1/3) = 13/3$$
$$x_2 = 4 + 2(1/3) = 14/3$$
$$x_3 = 4 + 3(1/3) = 5$$

$$\int_4^5 \sqrt{x^2+3}\,dx \approx T_3$$

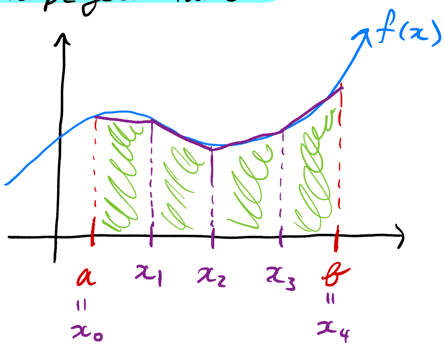
$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{1/3}{2}\left[\sqrt{4^2+3} + 2\sqrt{\left(\tfrac{13}{3}\right)^2+3} + 2\sqrt{\left(\tfrac{14}{3}\right)^2+3} + \sqrt{5^2+3}\right]$$

Lecture 33: Numerical integration

Recall: We computed $\int_a^b f(x)\,dx$ using the fundamental theorem of calculus (FTC), but we needed an antiderivative of $f(x)$.

Trapezoid rule



$n = \#$ of trapezoids

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$$

Recall area of trapezoid

$$\frac{b_1 + b_2}{2} h$$

$$\int_a^b f(x)\,dx \approx \frac{f(x_0)+f(x_1)}{2}\Delta x + \frac{f(x_1)+f(x_2)}{2}\Delta x + \frac{f(x_2)+f(x_3)}{2}\Delta x + \frac{f(x_3)+f(x_4)}{2}\Delta x$$

$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

In general, for any n we have

$$\int_a^b f(x)\,dx \approx T_n := \frac{\Delta x}{2}\Big[f(x_0) + \underbrace{2f(x_1) + \cdots + 2f(x_{n-1})}_{\text{multiplied by 2.}} + f(x_n)\Big]$$

e.g. ① Approximate $\int_3^5 7x^2+1\,dx$ using the trapezoid rule with $n=4$.

$f(x) = 7x^2+1$ $\Delta x = \frac{5-3}{4} = \frac{1}{2}$ $x_i = a + i\Delta x$
$= 3 + i(1/2)$

$x_0 = 3 + 0(1/2) = 3$
$x_1 = 3 + 1(1/2) = 7/2$ $x_3 = 3 + 3(1/2) = 9/2$
$x_2 = 3 + 2(1/2) = 4$ $x_4 = 3 + 4(1/2) = 5$

$$\int_3^5 7x^2+1\,dx \approx T_4 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1/2}{2}\left[7(3)^2+1 + 2\left(7\left(\tfrac{7}{2}\right)^2+1\right) + 2\left(7(4)^2+1\right) + 2\left(7\left(\tfrac{9}{2}\right)^2+1\right) + 7(5)^2+1\right]$$

② Approximate $\int_{-2}^{2} e^{x^2}\,dx$ using the trapezoid rule with $n=4$.

$f(x) = e^{x^2}$ $\Delta x = \frac{2-(-2)}{4} = \frac{4}{4} = 1$

$x_i = a + i\Delta x$
$= -2 + i$

$x_0 = -2 + 0 = -2$
$x_1 = -2 + 1 = -1$ $x_3 = -2 + 3 = 1$
$x_2 = -2 + 2 = 0$ $x_4 = -2 + 4 = 2$

$$\int_{-2}^{2} e^{x^2}\,dx \approx T_4 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right]$$

$$= \frac{1}{2}\left[e^{(-2)^2} + 2e^{(-1)^2} + 2e^{0^2} + 2e^{1^2} + e^{2^2} \right]$$

$$= \frac{1}{2}\left[e^4 + 2e + 2 + 2e + e^4 \right]$$

$$= \frac{1}{2}\left[2e^4 + 4e + 2 \right]$$

$$= e^4 + 2e + 1$$