Lecture 34: Exponential growth

Given the differential equation $\frac{dy}{dt} = 2y$, then how do we solve for y?

Guess: • $y = e^t$ $\qquad \frac{dy}{dt} = e^t = y$ $\qquad$ close!

• $y = e^{2t}$ $\qquad \frac{dy}{dt} = 2e^{2t} = 2y$ ✓

• $y = ce^{2t}$ $\qquad \frac{dy}{dt} = 2ce^{2t} = 2y$ ✓

All solutions to $\frac{dy}{dt} = 2y$ are of the form $y = ce^{2t}$ (where c is a constant).

In general given the differential equation $\frac{dy}{dt} = ky$ all the solutions are of the form

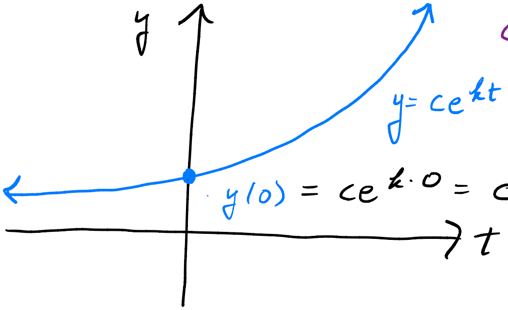
$$y = ce^{kt}$$

where c is some constant.

c : initial value of y

k : proportionality constant / growth rate

Exponential growth when c and $k > 0$ we say $y = ce^{kt}$ is an exponential growth model.



$c, k > 0$

e.g. ① A pop. of bacteria, $P(t)$ where t is time in days, is growing proportional to the population itself and the growth rate is 0.2. If the initial population is 40, what is the population after 40 days?

$$\frac{dP}{dt} = 0.2\,P \qquad P(0) = 40 \qquad P(40) = ??$$

$$P = ce^{0.2t} \qquad 40 = P(0) = ce^{0.2\cdot 0} = c$$

$$P = 40e^{0.2\cdot t}$$

$$P(40) = 40e^{0.2\cdot 40} \approx 119238.32 \text{ Bacteria}$$

(2) The rate of change of a pop. of a town is $\frac{dP}{dt} = kP$, where t is time in years.

If at $t = 3$ $P = 30,000$ and
at $t = 5$ $P = 40,000$, then
what is the pop. when $t = 10$?

$P = ce^{kt}$ $\qquad P(10) = ??$

what is c and k?

$$30,000 = P(3) = ce^{3k}$$
$$40,000 = P(5) = ce^{5k}$$

$$\frac{3}{4} = \frac{30,000}{40,000} = \frac{ce^{3k}}{ce^{5k}} = e^{-2k}$$

$$e^{-2k} = 3/4$$
$$\ln(e^{-2k}) = \ln(3/4)$$
$$-2k = \ln(3/4)$$
$$k = -\tfrac{1}{2}\ln(3/4) = \ln\left[(3/4)^{-1/2}\right]$$
$$= \ln(2/\sqrt{3}) > 0$$

$$30,000 = ce^{\ln(2/\sqrt{3})\cdot 3}$$
$$c = \frac{30,000}{e^{\ln[(2/\sqrt{3})^3]}} = \frac{30,000}{(2/\sqrt{3})^3}$$

$$P(t) = \frac{30{,}000}{(2/\sqrt{3})^3} e^{\ln(2/\sqrt{3}) \cdot t}$$

$$P(10) = \frac{30{,}000}{(2/\sqrt{3})^3} e^{\ln(2/\sqrt{3}) \cdot 10}$$

$$\approx 82112 \text{ people.}$$

(3) Jessica deposited \$40,000 into a savings account with interest compounding continuously. The annual rate of interest is 3%. How much money does she have in the account 12 years from now?

P = \$ in Jessica's account

t = time in years

3% → 0.03

$$\frac{dP}{dt} = 0.03P$$

$c = P(0) = 40{,}000.$

Goal: find $P(12) = ??$

Lecture 34: Exponential growth

Given the differential equation $\frac{dy}{dt} = 2y$, then how do we solve for y?

guesses :
- $y = e^t$ $\qquad \frac{dy}{dt} = e^t = y$ close
- $y = e^{2t}$ $\qquad \frac{dy}{dt} = 2e^{2t} = 2y$ ✓
- $y = ce^{2t}$ where c is a constant ✓

$$\frac{dy}{dt} = 2ce^{2t} = 2y$$

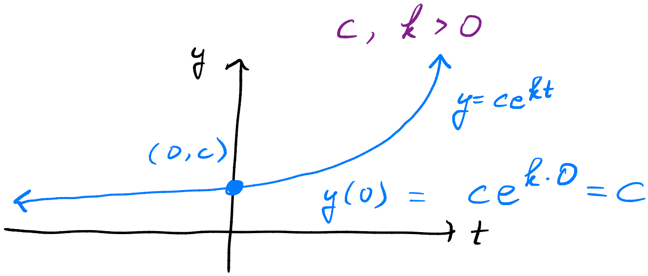
All solutions to $\frac{dy}{dt} = 2y$ are of the form $y = ce^{2t}$ where c is a constant.

In general the differential equation $\frac{dy}{dt} = ky$ has all solutions of the form $y = ce^{kt}$ where c is some constant.

c : initial value of y

k : proportionality constant / growth rate.

if c and k are greater than zero, then we say $y = ce^{kt}$ is an exponential growth model.



e.g. ① A population of Bacteria, $P(t)$ where t is time in days, is growing proportional to the population itself and the growth rate is 0.2. If the initial pop. is 40, then what is the pop. after 40 days?

$$\frac{dP}{dt} = 0.2\,P \qquad c = P(0) = 40 \qquad P(40) = ??$$

$$P = ce^{0.2t} = 40e^{0.2t}$$

$$P(40) = 40e^{0.2 \cdot 40} \approx 119238.32 \text{ Bacteria}$$

(2) The rate of change of a population of a town is $\underbrace{\frac{dP}{dt} = kP}$, where t is time in years.

If at $t = 3$ $P = 30{,}000$ and at $t = 5$ $P = 40{,}000$, then what is the population when $t = 10$?

$\rightarrow P = ce^{kt}$ $\qquad P(10) = ??$

$$30{,}000 = P(3) = ce^{3k}$$
$$40{,}000 = P(5) = ce^{5k}$$

$$\frac{3}{4} = \frac{30{,}000}{40{,}000} = \frac{ce^{3k}}{ce^{5k}} = e^{-2k}$$

$$e^{-2k} = 3/4$$
$$\ln(e^{-2k}) = \ln(3/4)$$
$$-2k = \ln(3/4)$$
$$k = -\tfrac{1}{2}\ln(3/4) = \ln\left[(3/4)^{-1/2}\right]$$
$$= \ln\left(2/\sqrt{3}\right) > 0$$

$$30{,}000 = ce^{\ln(2/\sqrt{3})\cdot 3}$$
$$30{,}000 = ce^{\ln[(2/\sqrt{3})^3]}$$
$$30{,}000 = c\,(2/\sqrt{3})^3$$
$$c = \frac{30{,}000}{(2/\sqrt{3})^3}$$

$$P(t) = \frac{30{,}000}{(2/\sqrt{3})^3}\,e^{\ln(2/\sqrt{3})t}$$ $\qquad P(10) \approx 82112$ people.

3. Jessica deposited \$40,000 into a savings account with interest compounded continuously. The annual rate of interest is 3%. How much money does she have in her account 12 years from now?

P = \$ in Jessica's account. $3\% \to 0.03$

t = years

$$\frac{dP}{dt} = 0.03P \qquad c = P(0) = 40{,}000$$

Goal is to find $P(12) = ??$