Lecture 35: Exponential decay

Exponential decay model

$\frac{dy}{dt} = ky$ the solutions are of the form $y = ce^{kt}$

if $k < 0$, this is called an exponential decay model.

e.g. ① The radio active isotope ^{226}Ra has a half life of about 1599 years. There are 85g of ^{226}Ra now, how much will remain after 1400 years?

half life follows exponential decay

$\rightsquigarrow P(t) = ce^{kt}$

P = grams of ^{226}Ra
t = time in years

$$85 = P(0) = ce^{k \cdot 0} = c$$

$$P(t) = 85e^{kt}$$

$$\frac{85}{2} = P(1599) = 85e^{k \cdot 1599}$$

$$\frac{85}{2} = 85e^{k \cdot 1599}$$

$$\frac{1}{2} = e^{1599k}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{1599k}\right)$$

$$-\ln 2 = 1599k$$

$$k = \frac{-\ln 2}{1599} < 0$$

$$P(1400) = 85\, e^{-\frac{\ln 2}{1599}\cdot 1400} \approx 46.33\text{ g of } {}^{226}Ra \checkmark$$

2) The radioactive isotope ${}^{239}Pu$ has a half life of about 24100 years.
After 1500 years there are 5g of ${}^{239}Pu$.

a) What is the initial quantity of ${}^{239}Pu$?

Want $P(0) = ce^{k\cdot 0} = c$

$P(t) = ce^{kt}$ $\quad$ P = ammount of ${}^{239}Pu$

t = time in years

$$\frac{c}{2} = P(24100) = ce^{24100k}$$

$$\frac{1}{2} = e^{24100k}$$

$$\ln(1/2) = \ln\left(e^{24100k}\right)$$

$$-\ln 2 = 24100k$$

$$k = \frac{-\ln 2}{24100}$$

$$5 = P(1500) = ce^{-\frac{\ln 2}{24100}\cdot 1500}$$

$$c = \frac{5}{e^{-\frac{\ln 2}{24100}\cdot 1500}} = 5\cdot 2^{15/241} \approx 5.22\text{g of } {}^{239}Pu.$$

③ A population of fish, P, is decreasing at a rate proportional to itself. If $P = 100{,}000$ when $t = 2$ years and $P = 50{,}000$ when $t = 5$ years, then what is pop. when $t = 10$ years?

$$\frac{dP}{dt} = kP \xrightarrow{\text{soln}} P(t) = ce^{kt} \qquad P(10) = ??$$

$$50{,}000 = P(5) = ce^{5k}$$
$$100{,}000 = P(2) = ce^{2k}$$

$$\frac{1}{2} = \frac{50{,}000}{100{,}000} = \frac{ce^{5k}}{ce^{2k}} = e^{3k}$$

$$\ln(1/2) = \ln(e^{3k})$$
$$-\ln 2 = 3k$$
$$k = -\frac{\ln 2}{3} < 0$$

$$50{,}000 = ce^{-\frac{\ln 2}{3}\cdot 5}$$

$$c = \frac{50{,}000}{e^{-\frac{\ln 2}{3}\cdot 5}} = 50000 \cdot 2^{5/3}$$

$$P(10) = 50{,}000 \cdot 2^{5/3} e^{-\frac{\ln 2}{3}\cdot 10} \approx 15{,}749.01 \text{ fish}$$

Lecture 35: Exponential decay

Exponential decay model

$$\frac{dy}{dt} = ky \xrightarrow{\text{soln}} y = ce^{kt} \qquad c \text{ a constant}$$

if $k < 0$, then this is an exponential decay model.

e.g. ① A pop. of fish, $P(t)$, is decreasing at a rate proportional to itself (t time in years).
If $P = 100{,}000$ at year 2 and
$P = 50{,}000$ at year 5, then
what is pop. of fish at year 10?

$$\frac{dP}{dt} = kP \xrightarrow{\text{soln}} P(t) = ce^{kt} \qquad P(10) = ??$$

$$50{,}000 = P(5) = ce^{5k}$$
$$100{,}000 = P(2) = ce^{2k}$$

$$\frac{1}{2} = \frac{50{,}000}{100{,}000} = \frac{ce^{5k}}{ce^{2k}} = e^{3k}$$

$$e^{3k} = 1/2$$
$$\ln(e^{3k}) = \ln(1/2)$$
$$3k = -\ln 2$$
$$k = -\frac{\ln 2}{3} < 0$$

$$50{,}000 = c\, e^{-\frac{\ln 2}{3}\cdot 5}$$

$$c = 50{,}000\, e^{5\cdot\frac{\ln 2}{3}}$$
$$= 50{,}000 \cdot 2^{5/3}$$

$$P(10) = 50000\cdot 2^{5/3}\, e^{-\frac{\ln 2}{3}\cdot 10} \approx 15749.01 \text{ fish.} \checkmark$$

(2) The radio active isotope ^{226}Ra has a half life of about 1599 years. There are 85g of ^{226}Ra now, how much remains after 1400 years?

half life follows exponential decay model.

P = ammount of ^{226}Ra
t = time in years

$$\frac{dP}{dt} = kP \rightsquigarrow P(t) = ce^{kt}. \qquad P(1400) = ??$$

$$85 = P(0) = ce^{k\cdot 0} = c$$

$$\frac{85}{2} = P(1599) = 85\, e^{1599k}$$

$$\frac{1}{2} = e^{1599k}$$

$$k = -\frac{\ln 2}{1599} < 0$$

$$P(1400) = 85\, e^{-\frac{\ln 2}{1599}\cdot 1400} \approx 46.33 \text{ g of } ^{226}Ra.$$

(3) The radioactive isotope ^{239}Pu has a half life of about 24,100 years. After 1500 years 5g of ^{239}Pu remain. What is the initial quantity of ^{239}Pu?

P = ammount of ^{239}Pu

t = time in years

$$\frac{dP}{dt} = kP \rightsquigarrow P(t) = ce^{kt}$$

goal: find $P(0) = ce^{k \cdot 0} = c$

$$\frac{c}{2} = P(24,100) = ce^{24,100k}$$

$$\frac{1}{2} = e^{24,100k}$$

$$\ln(1/2) = \ln(e^{24100k})$$

$$-\ln 2 = 24,100k$$

$$k = \frac{-\ln 2}{24,100} < 0$$

$$5 = P(1500) = ce^{\frac{-\ln 2}{24,100} \cdot 1500}$$

$$c = 5e^{\frac{\ln 2}{24,100} \cdot 1500} \approx 5.22 \text{ g of } ^{239}Pu$$