Lecture 9: The quotient rule; Derivatives of other trig functions.

HW 8 # 4) $y = 4e^x \sin x - 11 e^x \cos x$; find y'.

$$y' = \frac{d}{dx}[4e^x \sin x] + \frac{d}{dx}[-11 e^x \cos x]$$
$$= 4e^x \sin x + 4e^x \cos x - 11e^x \cos x - 11e^x(-\sin x)$$
$$= 15e^x \sin x - 7e^x \cos x \checkmark$$

Quotient rule:

$$\frac{d}{dx}\left[\frac{g(x)}{h(x)}\right] = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{(\,h(x)\,)^2}$$

e.g. ① $f(x) = \dfrac{x^3}{x^2+3x+1}$ $\begin{matrix} = g(x) \\ = h(x) \end{matrix}$; find $f'(x)$

$$f'(x) = \frac{d}{dx}\left[\frac{x^3}{x^2+3x+1}\right] = \frac{(x^3)'(x^2+3x+1) - x^3(x^2+3x+1)'}{(x^2+3x+1)^2}$$

$$= \frac{3x^2(x^2+3x+1) - x^3(2x+3+0)}{(x^2+3x+1)^2}$$

$$= \frac{3x^2(x^2+3x+1) - x^3(2x+3)}{(x^2+3x+1)^2}$$

② $f(x) = \tan x$ find $f'(x)$.

$$f(x) = \frac{\sin x}{\cos x}$$

$$f'(x) = \frac{d}{dx}\left[\tan x\right] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

$$= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

Derivatives of trig functions

$$\frac{d}{dx}\left[\tan x\right] = \sec^2 x \qquad \frac{d}{dx}\left[\cot x\right] = -\csc^2 x$$

$$\frac{d}{dx}\left[\sec x\right] = \sec x \tan x \qquad \frac{d}{dx}\left[\csc x\right] = -\csc x \cot x$$

↑↑ all applications of the quotient rule ↑↑

e.g. 3 $y = -3e^{x}\sec x$; $dy/dx = ?$

$$\frac{dy}{dx} = \frac{d}{dx}\left[-3e^{x}\sec x\right] \text{ product rule.}$$

$$= -3e^{x}\sec x + -3e^{x}\sec x\tan x$$

4 $f(x) = \dfrac{x^2 + \sqrt[3]{x}}{1-x}$; $df/dx = ?$

$$\frac{df}{dx} = \frac{d}{dx}\left[\frac{x^2 + x^{1/3}}{1-x}\right]$$

$$= \frac{(x^2 + x^{1/3})'(1-x) - (x^2 + x^{1/3})(1-x)'}{(1-x)^2}$$

$$= \frac{(2x + \frac{1}{3}x^{-2/3})(1-x) - (x^2 + x^{1/3})(-1)}{(1-x)^2}$$

$$= \frac{(2x + \frac{1}{3}x^{-2/3})(1-x) + (x^2 + x^{1/3})}{(1-x)^2}$$

5. Let a be a constant $(a \neq 0)$

Let $f(x) = \dfrac{a + x^2}{a \sin x}$; find $\left.\dfrac{df}{dx}\right|_{x=\frac{\pi}{2}}$

$$\frac{df}{dx} = \frac{d}{dx}\left[\frac{a + x^2}{a \sin x}\right] = \frac{(a+x^2)'(a\sin x) - (a+x^2)(a \sin x)'}{(a \sin x)^2}$$

$$\frac{d}{dx}\left[a + x^2\right] = \frac{d}{dx}[a] + \frac{d}{dx}[x^2]$$

$$= \quad 0 \quad + \quad 2x$$

$$\frac{d}{dx}\left[a \sin x\right] = a \frac{d}{dx}[\sin x] = a \cos x$$

$$\frac{df}{dx} = \frac{2x(a\sin x) - (a + x^2)(a \cos x)}{(a \sin x)^2}$$

$$\left.\frac{df}{dx}\right|_{x=\frac{\pi}{2}} \left(= f'\left(\tfrac{\pi}{2}\right)\right) = \frac{2\left(\frac{\pi}{2}\right) \cdot a \cdot \sin\left(\frac{\pi}{2}\right) - \left(a + \left(\frac{\pi}{2}\right)^2\right) \cdot a \cos\left(\frac{\pi}{2}\right)}{\left(a \sin(\pi/2)\right)^2}$$

$$= \frac{\pi \cdot a \cdot 1 - (a + \pi^2/4) \cdot a \cdot 0}{(a \cdot 1)^2}$$

$$= \frac{a\pi}{a^2}$$

$$= \frac{\pi}{a}$$

Lecture 9: The quotient rule; Derivatives of other trig functions

HW 8 # 9 | $y = 9x^2 e^x$; find all x values such that y has a horizontal tangent line.

when does $y'(x) = 0$?

$$y'(x) = \frac{d}{dx}\left[9x^2e^x\right] = 18xe^x + 9x^2e^x$$

$$\begin{aligned} 0 &= 18xe^x + 9x^2e^x \\ &= 9xe^x(2+x) \end{aligned}$$

$$9xe^x = 0 \qquad \text{or} \qquad 2 + x = 0$$

$$9x = 0 \qquad\qquad x = -2$$

$$x = 0$$

Quotient rule:

$$\frac{d}{dx}\left[\frac{g(x)}{h(x)}\right] = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$$

e.g. ① $f(x) = \dfrac{x^3}{x^2+3x+1}$; find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\frac{x^3}{x^2+3x+1}\right] = \frac{(x^3)'(x^2+3x+1) - x^3(x^2+3x+1)'}{(x^2+3x+1)^2}$$

$$= \frac{3x^2(x^2+3x+1) - x^3(2x+3+0)}{(x^2+3x+1)^2}$$

$$= \frac{3x^2(x^2+3x+1) - x^3(2x+3)}{(x^2+3x+1)^2}$$

② $f(x) = \tan x$; find $f'(x)$.

$$f(x) = \frac{\sin x}{\cos x}$$

$$f'(x) = \frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

$$= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

Derivatives of other trig functions

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

↑↑ all of these are applications of quotient rule ↑↑

③ $y = -3e^{x}\sec x$; $dy/dx = ?$.

$$\frac{dy}{dx} = \frac{d}{dx}\left[-3e^{x}\sec x\right]$$
$$= (-3e^{x})'\sec x + (-3e^{x})(\sec x)'$$
$$= -3e^{x}\sec x - 3e^{x}\sec x\tan x \qquad \checkmark$$

④ $f(x) = \dfrac{x^2 + \sqrt[3]{x}}{1-x}$; find $\dfrac{df}{dx}$.

$$\frac{df}{dx} = \frac{d}{dx}\left[\frac{x^2 + x^{1/3}}{1-x}\right] = \frac{(x^2 + x^{1/3})'(1-x) - (x^2 + x^{1/3})(1-x)'}{(1-x)^2}$$

$$= \frac{(2x + \frac{1}{3}x^{-2/3})(1-x) - (x^2 + x^{1/3})(0-1)}{(1-x)^2}$$

$$= \frac{(2x + \frac{1}{3}x^{-2/3})(1-x) + (x^2 + x^{1/3})}{(1-x)^2} \qquad \checkmark$$

⑤ Let a be a constant and $(a \neq 0)$

$$f(x) = \frac{a + x^2}{a \sin x} \quad ; \quad \left.\frac{df}{dx}\right|_{x=\pi/2} = ? \, .$$

$$\frac{df}{dx} = \frac{d}{dx}\left[\frac{a + x^2}{a \sin x}\right] = \frac{(a + x^2)'(a \sin x) - (a + x^2)(a \sin x)'}{(a \sin x)^2}$$

$$\frac{d}{dx}[a + x^2] = \frac{d}{dx}[a] + \frac{d}{dx}[x^2]$$

$$= 0 + 2x$$

$$\frac{d}{dx}[a \sin x] = a \frac{d}{dx}[\sin x] = a \cos x$$

$$\frac{df}{dx} = \frac{2x(a \sin x) - (a + x^2) \cdot a \cos x}{(a \sin x)^2}$$

$$\left.\frac{df}{dx}\right|_{x=\frac{\pi}{2}} \left(= f'\left(\frac{\pi}{2}\right)\right) = \frac{2\left(\frac{\pi}{2}\right) \cdot a \sin\left(\frac{\pi}{2}\right) - \left(a + \frac{\pi^2}{4}\right) \cdot a \cos\frac{\pi}{2}}{\left(a \sin\frac{\pi}{2}\right)^2}$$

$$= \frac{\pi \cdot a \cdot 1 - (a + \pi^2/4) \cdot a \cdot 0}{(a \cdot 1)^2}$$

$$= \frac{a\pi}{a^2}$$

$$= \frac{\pi}{a}$$