HW 34 #4

$$x' = \underbrace{\begin{pmatrix} 3 & -15 & 0 & -5 \\ 0 & 3 & 0 & 0 \\ 10 & -30 & -2 & -10 \\ 0 & -15 & 0 & -2 \end{pmatrix}}_{A} x$$

Step 1

Eigenvalues:

$$0 = \det(A - \lambda I)$$

$$= \begin{vmatrix} 3-\lambda^{+} & -15^{-} & 0^{+} & -5 \\ 0 & 3-\lambda & 0^{-} & 0 \\ \cancel{10} & \cancel{-30} & -2-\lambda^{+} & \cancel{-10} \\ 0 & -15 & 0 & -2-\lambda \end{vmatrix}$$

$$= (1)(-2-\lambda) \begin{vmatrix} 3-\lambda^{+} & \cancel{-15} & \cancel{-5} \\ \cancel{0} & 3-\lambda & 0 \\ \cancel{0} & -15 & -2-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(3-\lambda) \begin{vmatrix} 3-\lambda & 0 \\ -15 & -2-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(3-\lambda)(3-\lambda)(-2-\lambda)$$

$$0 = (\lambda+2)^2(\lambda-3)^2$$

$\lambda = -2$ or 3 are the eigenvalues.

Step 2: find Eigen vectors (or generalized e.v.)

$\lambda = 3$ | Compute a basis for the null space of $A - 3I$.

$$A - 3I = \begin{pmatrix} 0 & -15 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 10 & -30 & -5 & -10 \\ 0 & -15 & 0 & -5 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(A - 3I)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\begin{aligned} 2a - c &= 0 \\ 3b + d &= 0 \end{aligned}$$

$$\left.\begin{aligned} c &= s \\ d &= t \end{aligned}\right\} \text{free}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = s\begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$$ Basis for the null space is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -3 \end{pmatrix} \right\}$

Since the multiplicity of $\lambda = 3$ is 2 and we got 2 linearly ind. e.v. we are done.

$\underline{\lambda = -2}$ Repeat above process

$$A - (-2)I = A + 2I = \begin{pmatrix} 5 & -15 & 0 & -5 \\ 0 & 5 & 0 & 0 \\ 10 & -30 & 0 & -10 \\ 0 & -15 & 0 & 0 \end{pmatrix}$$

Goal is to compute a basis for the nullspace of $A + 2I$.

We hope to get 2 lin. ind. e.v. since the mult. of $\lambda = -2$ is 2.

a basis for the nullspace is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$x(t) = c_1 e^{3t}\begin{pmatrix}1\\0\\2\\0\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}0\\1\\0\\-3\end{pmatrix} + c_3 e^{-2t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + c_4 e^{-2t}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

HW 35 #2 Eigenvalues: $\lambda_1 = 2 \quad \lambda_2 = -2$

- Both Eigenvalues are real
- distinct
- opposite signs

$\Rightarrow$ Saddle point.

HW 34 #3

$$x' = \underbrace{\begin{pmatrix} -10 & -9 & 0 \\ 1 & -4 & 0 \\ 1 & 3 & -7 \end{pmatrix}}_{=A} x$$

Step 1: Compute characteristic eq. / find Eigenvalues.

$$0 = \det(A - \lambda I)$$

$$= \begin{vmatrix} -10-\lambda^{+} & -9^{-} & 0^{+} \\ 1 & -4-\lambda & 0^{-} \\ 1 & 3 & -7-\lambda^{+} \end{vmatrix}$$

$$= (1)(-7-\lambda) \begin{vmatrix} -10-\lambda & -9 \\ 1 & -4-\lambda \end{vmatrix}$$

$$= (-7-\lambda)\left((-10-\lambda)(-4-\lambda) + 9\right)$$

$$= -(\lambda+7)(\lambda^2 + 14\lambda + 49)$$

$$= -(\lambda+7)(\lambda+7)^2$$

$$0 = -(\lambda+7)^3$$

$\lambda = -7$ multiplicity of 3

Step 2: Compute Eigenspace.

$$\{\text{Eigenspace of } \lambda = -7\} = \{\text{nullspace of } A-(-7)I\}$$

Goal: compute a basis for the nullspace of $A+7I$

$$\text{nullspace of } A+7I = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; (A+7I)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$A+7I = \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} a + 3b &= 0 \\ b &= s \\ c &= t \end{aligned} \Bigg\} \text{ free} \qquad \begin{aligned} a &= -3s \\ b &= s \\ c &= t \end{aligned}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = s\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

basis $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Step 3: Generalized Eigenvectors

$$(A+7I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (A + 7I)^2 v_2 = 0$$

$$v_1 = (A + 7I) v_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$x(t) = c_1 e^{-7t} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-7t} v_1 + c_3 e^{-7t} \left[v_1 t + v_2 \right]$$