Spring 2019 #3

$$(e^x \sin y - 2x - y) + (e^x \cos y - x + 3y^2)\frac{dy}{dx} = 0 \qquad y(0) = \pi$$

$$(\underbrace{e^x \sin y - 2x - y}_{M})\,dx + (\underbrace{e^x \cos y - x + 3y^2}_{N})\,dy = 0$$

To be exact $M_y = N_x$

$$e^x \cos y - 1 = e^x \cos y - 1$$

$$\int M\,dx = \int e^x \sin y - 2x - y \, dx$$
$$= e^x \sin y - x^2 - xy + c(y)$$

$$\int N\,dy = \int e^x \cos y - x + 3y^2 \, dy$$
$$= e^x \sin y - xy + y^3 + c(x)$$

$$f(x,y) = e^x \sin y - xy - x^2 + y^3 + c = 0$$

$$f(0,\pi) = e^0 \sin(\pi) - 0\cdot\pi - 0^2 + \pi^3 + c = 0$$
$$0 = \pi^3 + c$$
$$c = -\pi^3$$

E. $e^x \sin y - x^2 - xy + y^3 = \pi^3$

Spring 2016 #3 $\qquad \frac{dy}{dx} = \frac{y}{x(\ln x)^2} \qquad y(e) = e$

$$\int \frac{dy}{y} = \int \frac{dx}{x(\ln x)^2}$$

$u = \ln x$

$du = \frac{dx}{x}$

$$\ln|y| = \int \frac{du}{u^2}$$

$$\ln|y| = \frac{-1}{u} + c$$

$$\ln|y| = \frac{-1}{\ln x} + c$$

$$\ln|y(e)| = \frac{-1}{\ln e} + c$$

$$\ln e = -1 + C$$

$$1 = -1 + c$$

$$c = 2$$

$$\ln|y| = \frac{-1}{\ln x} + 2$$

$$\ln(y(e^2)) = ?$$

$$\ln(y(e^2)) = \frac{-1}{\ln e^2} + 2 = -\frac{1}{2} + 2 = \frac{3}{2}$$

D.

① $W = \{ A \ 3 \times 3$ matrix w/ real entries such that $\det A = 1 \}$

Is W a subspace of $M^{2\times 2}(\mathbb{R})$ (2×2 matrices w/ real entries).

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is in W

$2I = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad \det(2I) = 8 \neq 1$

if w, v are vectors in W, then
- $w + v$ to be in W
- $c \cdot w$ to be in W (where c real #)

$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \ x + 2y - z = 0 \quad x, y, z \text{ are in } \mathbb{R} \right\}$

Is W a subspace of $\mathbb{R}^3$?

$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ in W. $\quad c$ in $\mathbb{R}$.

$\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$ is this in W?

$$(x_1 + x_2) + 2(y_1 + y_2) - (z_1 + z_2)$$
$$= (x_1 + 2y_1 - z_1) + (x_2 + 2y_2 - z_2)$$

$= 0 + 0 = $

$\begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix}$ is this in W? Yes

$x_1 + 2y_1 - z_1 = 0$

$\times c$

$cx_1 + 2cy_1 - cz_1 = 0$

$\Rightarrow$ Yes $\begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix}$ is in W

W is a subspace of $\mathbb{R}^3$.

Undetermined Coeff. $x'' - 3x' + 2x = e^t$

find the particular solution.

$$x'' - 3x' + 2x = 0$$
$$r^2 - 3r + 2 = 0$$
$$(r-2)(r-1) = 0$$
$$r = 1, 2$$

homogeneous soln: $x_h(t) = c_1 e^t + c_2 e^{2t}$

Particular soln will involve the derivatives of e^t

$$x_p = Ate^t$$
$$x_p' = Ae^t + Ate^t$$
$$x_p'' = Ae^t + Ae^t + Ate^t$$
$$= 2Ae^t + Ate^t$$

$$x'' - 3x' + 2x = e^t$$
$$x_p'' - 3x_p' + 2x_p = e^t$$
$$(\underline{2Ae^t} + \underline{Ate^t}) - 3(\underline{Ae^t} + \underline{Ate^t}) + \underline{2Ate^t} = e^t$$
$$(2A - 3A)e^t + (A - 3A + 2A)te^t = e^t$$
$$-Ae^t = e^t$$
$$-A = 1$$
$$A = -1$$

$$x_p = -te^t$$

Spring 2019 24 | Let $x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be a soln to

$$x' = \underbrace{\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}}_{A} x \qquad x(0) = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

Find x.

Eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} \\ &= (1-\lambda)(4-\lambda) + 2 \\ &= 4 - 5\lambda + \lambda^2 + 2 \\ &= \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 3)(\lambda - 2) \end{aligned}$$

$$\lambda = 3, 2$$

Eigenspace of $\lambda = 3$.

$$(A - 3I)v = 0$$

$$A - 3I = \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$v = \begin{pmatrix} a \\ b \end{pmatrix} \implies \begin{aligned} 2a + b &= 0 \\ a &= t \end{aligned} \qquad b = -2a = -2t$$

$$v = \begin{pmatrix} t \\ -2t \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Eigenspace for $\lambda = 2$

$$A - 2I = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \qquad \underline{(A-2I)v} = 0$$

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$v = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{array}{l} a + b = 0 \\ b = t \end{array} \quad \Rightarrow \quad a = -b = -t$$

$$v = \begin{pmatrix} -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

General soln: $\quad x(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Recall $\quad x(0) = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix} = x(0) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 \\ -2c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & -1 & -4 \\ -2 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -4 \\ 0 & -1 & -7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -4 \\ 0 & 1 & 7 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 7 \end{array}\right) \qquad \begin{array}{l} c_1 = 3 \\ c_2 = 7 \end{array}$$

$$x(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= 3e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 7e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{3t} - 7e^{2t} \\ -6e^{3t} + 7e^{2t} \end{pmatrix}$$

$$x(1) = \begin{pmatrix} 3e^{3} - 7e^{2} \\ -6e^{3} + 7e^{2} \end{pmatrix}$$ B.

Spring 2019 10) $t^2 y'' + 2t y' - 2y = 0 \quad t > 0$

Given $y_1(t) = t$ is a soln

Find a second solution to the ode.

$y_2 = y_1 v = t v$ $\quad$ v is a fun. of t

$y_2' = v + t v'$

$y_2'' = v' + v' + t v''$

$= 2v' + t v''$

$$t^2 (2v' + t v'') + 2t(v + t v') - 2tv = 0$$

$$\Downarrow$$

$$t^3 v'' + t^2 v' = 0$$

Let $w = v'$ $\rightsquigarrow$ $t^3 w' + t^2 w = 0$

then use integrating factor

See Notes in Rec 8 on website for full solution to similar prob.