HW 24 #7 $\qquad x^2y'' - xy' - 15y = 0 \qquad (x>0)$

$y_1 = x^5$ is a solution.

Let $y_2(x) = y_1(x)\,v(x)$ be another solution.
$= x^5 v(x)$

$$y_2' = 5x^4 v + x^5 v'$$
$$y_2'' = 20x^3 v + 5x^4 v' + 5x^4 v' + x^5 v''$$
$$= 20x^3 v + 10x^4 v' + x^5 v''$$

$$x^2 y_2'' - x y_2' - 15 y_2 = 0$$
$$x^2(20x^3 v + 10x^4 v' + x^5 v'') - x(5x^4 v + x^5 v') - 15x^5 v = 0$$
$$20x^5 v + 10x^6 v' + x^7 v'' - 5x^5 v - x^6 v' - 15x^5 v = 0$$

(divide by x^5

$$\cancel{20v} + 10xv' + x^2 v'' \cancel{- 5v} - xv' \cancel{- 15v} = 0$$

$$x^2 v'' + 9xv' = 0$$

let $w = v'$, then $\frac{dw}{dx} = w' = v''$

$$x^2 \frac{dw}{dx} + 9xw = 0$$
$$x^2 \frac{dw}{dx} = -9xw$$
$$\int \frac{dw}{w} = \int \frac{-9}{x}\,dx$$
$$\ln w = -9 \ln x + C_1$$
$$v' = w = e^{C_1} e^{\ln x^{-9}} = e^{C_1} x^{-9}$$

$$v = \int v' \, dx = \int e^{c_1} x^{-9}$$

$$= \frac{e^{c_1}}{-8} x^{-8} + c_2$$

choose c_1 and c_2 so $v = x^{-8}$

$$y_2(x) = x^5 v(x) = x^5 x^{-8} = x^{-3}$$

HW 24 #4 $f(x) = 13, \quad g(x) = 6x, \quad h(x) = 2x^2$

Use the Wronskian to determine if f, g, h are linearly independent.

$$W(f, g, h)(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

$$= \begin{vmatrix} 13 & 6x & 2x^2 \\ 0 & 6 & 4x \\ 0 & 0 & 4 \end{vmatrix}$$

$$= 13 \cdot 6 \cdot 4$$

$$= 312$$

f, g, h are linearly indpendent if and only if $W(f, g, h)(x) \neq 0$.

HW 24 #6 $\quad y^{(3)} + 9y' = 0; \quad y(0) = 2, \; y'(0) = -1, \; y''(0) = 3$

$y_1 = 1, \quad y_2 = \cos 3x \quad y_3 = \sin 3x$

are linearly independent solutions to the differential equation.

$$y(x) = c_1(1) + c_2(\cos 3x) + c_3(\sin 3x)$$
$$y' = 0 - 3c_2 \sin 3x + 3c_3 \cos 3x$$
$$y'' = -9c_2 \cos 3x - 9c_3 \sin 3x$$

$$c_1 + c_2 = 2 \qquad c_1 = 7/3$$
$$3c_3 = -1 \qquad c_3 = -1/3$$
$$-9c_2 = 3 \qquad c_2 = -1/3$$

$$y(x) = 7/3 - 1/3 \cos 3x - 1/3 \sin 3x$$

HW 24 #1 $\quad f(x) = 5x, \quad g(x) = 3x^2, \quad h(x) = 8x - 8x^2$

Finding a non trivial linear combination which is 0 everywhere.

$$(24)\, 5x + \left(\overset{c_1}{\text{some \#}}\right) 3x^2 + \left(\overset{c_2}{\text{some other \#}}\right)(8x - 8x^2) = 0$$

$$-120x = 3c_1x^2 + c_2(8x - 8x^2)$$
$$= 3c_1x^2 + 8c_2x - 8c_2x^2$$
$$= 8c_2x + (3c_1 - 8c_2)x^2$$

$$-120 = 8c_2 \qquad c_2 = -15$$

$$0 = 3c_1 - 8c_2 \qquad c_1 = -40$$