MA 262 Vector Spaces

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Definition 0.1. A *set* S is a collection of things called *elements*. We use the notation

$$S = \{r, s, t, \dots\}$$

to mean that S is a set and that its elements are r, s, t and so one.

Definition 0.2. Given two sets S and T we say that S is *subset* of T, denoted $S \subset T$, if every element of S is also an element of T.

Definition 0.3. Given two sets S and T a *function* $f : S \to T$ (read f is a function from S to T) assigns to every element s of S a unique element t of T. We denote this assignment $f(s) = t$.

1 Linear Spaces

Linear algebra is the study of linear transformations between vector spaces. In all that follows we will only be looking at the case when our vector spaces are defined over the real numbers, $\mathbb{R}$. One may (and should) replace $\mathbb{R}$ with any field, but for the sake of introducing the subject we will set full generality aside.

It is often standard to begin with the definition of a vector space, but for a student that is not familiar with abstract algebra the following definition will be giberish. The definition is included for the inquisitive reader, but it is recommended to simply skim over the words and immediately jump into examples.

Definition 1.1. A *vector space* (also called a linear space) is a set V, whose elements are called *vectors*, equiped with the operations of addition and scalar multiplication such that for any u, v and w in V and any real numbers a, b and c we have that

1. (closure) $u + v$ and $c \cdot v$ are a vectors in V;
2. (associativity) $(u + v) + w = u + (v + w)$;
3. (additive identity) there exists a zero vector $\vec{0}$ such that $\vec{0} + v = v = v + \vec{0}$;
4. (additive inverses) there exists a negative vector u^- such that $u + u^- = \vec{0} = u^- + u$;
5. (commutativity) $u + v = v + u$;
6. (scalar distributivity) $(a + b) \cdot u = a \cdot u + b \cdot u$;
7. (vector distributivity) $a \cdot (u + v) = a \cdot u + a \cdot v$;
8. (scalar identity) $1 \cdot u = u$;
9. and $(ab) \cdot u = a \cdot (b \cdot u)$.

We call the real numbers in this case *scalars*.

Remark 1. The given definition is actually what is called an $\mathbb{R}$-linear space. One may use any field for the scalars. Another common field used in engineering is the complex numbers, $\mathbb{C}$.

Example 1.2. When a person asks what is a vector the common answer is an object with direction and magnitude. In particular such a person is probably thinking about the following vector space. Let n be any positive whole number, then n-dimensional (real) space is the vector space

$$\mathbb{R}^n := \{(a_1, \dots, a_n) \text{ where } a_1, \dots, a_n \text{ are real numbers.}\}$$

where addition is defined by

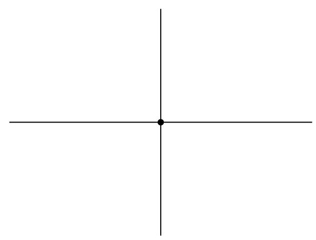
$$(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_n, \dots, a_n + b_n)$$

and scalar multiplication is defined by

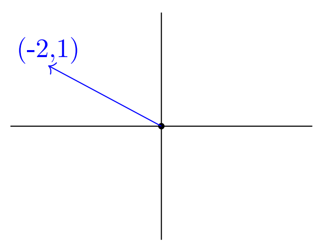
$$c \cdot (a_1, \dots, a_n) = (ca_1, \dots, ca_n).$$

The zero vector will then be $(0, \dots, 0)$, i.e., the origin.

If $n = 2$, then we get the real plane $\mathbb{R}^2$. Geometrically $\mathbb{R}^2$ corresponds to the following picture:



where the illustrated lines are the vertical and horizontal axis and the dot is the origin. A vector in $\mathbb{R}^2$ may be interpruted geometrically as well:



As you may guess if $n = 3$, then we get 3 dimensional space (like the one we live in) and if $n > 3$ we get higher dimensions! $\mathbb{R}^n$ will allow us to do geometry in dimensions larger than the one we live in. Often times in applications one will not be lucky enough to encounter a real life problem that only uses 3 dimensions. The real world, unlike math, is messy and often problems one encounters in practice will have a large handful of variables each of which induces a unique dimension.

Problem 1. Verify that $\mathbb{R}^2$ is indeed a vector space by checking each item in the definition.

Problem 2. Interprut the operations of addition and scalar multiplication geometrically for $\mathbb{R}^2$.

Problem 3. Draw a picture of $\mathbb{R}^3$.

Example 1.3. The next example shows that the slogan "a vector is an object with direction and magnitude" is misleading. A polynomial is a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0$ where $a_n, \dots, a_0$ are real numbers. We call a_k the k-th coefficient of f and we say the degree of f is n. Then the set

$$\mathbb{P} = \{\text{polynomials with real coefficients}\}$$

is a vector space when it is equiped with the usual notion of addition and scalar multiplication. In fact, the multiplication is defined for any two polynomials not just for a scalar and a polynomial. Such a structure is called a real algebra.

In this example the polynomial $g(x) = x^2 + 1$ is a vector in the vector space A, but it does not have any obvious direction or magnitude. If one pursues more linear algebra than will be offered in this course, then they will see that most notions of vector fail to have direction or magnitude.

Problem 4. Verify that $\mathbb{P}$ is a vector space (this should be nearly the same argument as for $\mathbb{R}^2$).

Example 1.4. In the previous example we saw that a certain collection of functions formed a vector space, namely that of polynomials. We also have the following vector spaces made of of functions $f(x)$ from $\mathbb{R}$ to $\mathbb{R}$:

$$\mathbb{R}^{\mathbb{R}} = \{\text{all functions } f : \mathbb{R} \to \mathbb{R}\}$$
$$\mathcal{C}^0 = \{\text{continuous functions } f : \mathbb{R} \to \mathbb{R}\}$$
$$\mathcal{C}^1 = \{\text{continuous functions } f : \mathbb{R} \to \mathbb{R} \text{ with continuous derivative}\}$$
$$\mathbb{P}(n) = \{\text{polynomials } f : \mathbb{R} \to \mathbb{R} \text{ with degree at most } n\}.$$

A simple refresher of the calculus definitions tells us that

$$\mathbb{P}(n) \subset \mathbb{P} \subset \mathcal{C}^1 \subset \mathcal{C}^0 \subset \mathbb{R}^{\mathbb{R}}.$$

In general vector spaces that are subsets of vector spaces are common and essential to our study.

Definition 1.5. Given a vector space V a *subspace* of V is a subset $W \subset V$ such that W is a vector space with respect to the addition and scalar multiplication given from V.

Example 1.6. *Claim:* $\mathbb{P}(1)$ is a subspace of $\mathbb{P}(2)$. *Proof:* Let $f(x) = a_1x + a_0$ and $g(x) = b_1x + b_0$ be vectors in $\mathbb{P}(1)$. Then we may view them as vectors in $\mathbb{P}(2)$ since $f(x) = 0x^2 + a_1x + a_0$ and $g(x) = 0x^2 + b_1x + b_0$. Now if we add f and g as vectors in $\mathbb{P}(2)$ we obtain

$$f(x) + g(x) = (0x^2 + a_1x + a_0) + (0x^2 + b_1x + b_0) = (0 + 0)x^2 + (a_1 + b_1)x + (a_0 + b_0)$$

which is precisely their sum in $\mathbb{P}(1)$, $(a_1+b_1)x+a_0+b_0$. Now let c be any real number then $c\cdot f(x) = ca_1x+ca_0$ in $\mathbb{P}(1)$. Since in $\mathbb{P}(2)$ we have

$$c \cdot f(x) = c(0x^2 + a_1x + a_0) = 0x^2 + ca_1x + ca_0$$

and $0x^2 + ca_1x + ca_0 = ca_1x + ca_0$, then we have that $\mathbb{P}(1)$ is a vector space with respect to the addition and scalar multiplication of $\mathbb{P}(2)$. Hence $\mathbb{P}(1)$ is a subspace of $\mathbb{P}(2)$.

Problem 5. Show that $\mathbb{R}^2$ may be thought of as a subspace of $\mathbb{R}^3$. Hint: we may consider a vector (a_1, a_2) in $\mathbb{R}^2$ as a vector in $\mathbb{R}^3$ by assigning it to $(a_1, a_2, 0)$.

Example 1.7. Let $m < n$ be positive whole numbers, then the following subsets are subspaces

$$\mathbb{P}(m) \subset \mathbb{P}(n) \subset \mathbb{P} \subset \mathcal{C}^1 \subset \mathcal{C}^0 \subset \mathbb{R}^{\mathbb{R}}$$

and

$$\mathbb{R}^m \subset \mathbb{R}^n.$$

Theorem 1.8. *Let V be a vector space and W a subset of V. Then W is a subspace of V if for all u and w in W and real number c we have that $u + c \cdot w$ is also in W.*

Proof: Let u and w be elements of W and $c = 1$, then $u + w = u + c \cdot w$ is W. Now let $u = \vec{0}$, then $c \cdot w = u + c \cdot w$ is in W.

2 Linear independence

Trying to work with vector spaces without a basis is akin to using a lighter to bake a pie, please just use the oven. For our purposes the "oven" will be what is called a basis. If one is given a vector space and a basis, any problem that is posed about the space can be quickly solved using the basis. A basis for a vector space is so important that the reader has certainly already encountered and used them in their math.

Let us fix a vector space V, then a basis $\mathcal{B}$ is a subset of V that should somehow "express" all the data found in V in a "minimal" way.

Definition 2.1. Let V be a vector space and $\{v_1, \ldots, v_n\}$ be a subset of V. If $a_1, \ldots, a_n$ are real numbers, then a linear combination of $v_1, \ldots, v_n$ is a vector of the form

$$a_1 \cdot v_1 + \cdots + a_n \cdot v_n.$$

Furthermore, if $a_1 = \cdots = a_n = 0$, then we say that this is a trivial combination.

Definition 2.2. Let V be a vector space and $\{v_1, \ldots, v_n\}$ be a subset of V. Then the span of $v_1, \ldots, v_n$ is the set of all linear combinations of $v_1, \ldots, v_n$, i.e.,

$$\text{span}\{v_1, \ldots, v_n\} = \{a_1 v_1 + \cdots + a_n v_n \text{ where } a_1, \ldots, a_n \text{ are real numbers.}\}$$

With this new language we have half of our definition of a basis, namely span. Worded another way, we require a basis to have the property that any vector in V can be expressed as a linear combination of vectors from the basis.

Example 2.3. Let us calculate the span of the following vectors in $\mathbb{R}^3$:

$$u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

A linear combination of u, v and w is any vector of $\mathbb{R}^3$ of the form

$$a \cdot u + b \cdot v + c \cdot w = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ 2b \\ 0 \end{pmatrix} + \begin{pmatrix} 3c \\ 2c \\ c \end{pmatrix} = \begin{pmatrix} a + 3c \\ 2b + 2c \\ a + c \end{pmatrix}$$

where a, b and c are real numbers. Thus

$$\text{span}\{u, v, w\} = \{ \begin{pmatrix} a + 3c \\ 2b + 2c \\ a + c \end{pmatrix} \text{ where } a, b, c \text{ are real numbers } \}.$$

Problem 6. Show that the span of the vectors in the previous example is all of $\mathbb{R}^3$.

Problem 7. Show that the vectors 1, x and x^2 span $\mathbb{P}(2)$.

Problem 8. Let V be a vector space, show that V spans V.

The second condition we required of a basis was for it to be "minimal". The previous problem shows that a spanning set always exists, namely the space itself, but we are after a spanning set that is as small as possible.

Definition 2.4. Let V be a vector space and let $v_1, \ldots, v_n$ be vectors in V, then we say that $v_1, \ldots, v_n$ are *linearly dependent* if the zero vector can be expressed as a nontrivial linear combination of $v_1, \ldots, v_n$, i.e., there exists $a_1, \ldots, a_n$ not all zero such that

$$\vec{0} = a_1 v_1 + \cdots + a_n v_n.$$

We say that $v_1, \ldots, v_n$ are *linearly independent* if they are not linearly dependent.

Linear indpendence is the correct way to interprut the minimality condition for a basis. The reason for this is as follows:

The following vectors of $\mathbb{R}^2$ span the entirety of $\mathbb{R}^2$

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

but they have a linear dependency since $\vec{0} = 2u - v - w$. If we solve for w, then we obtain $w = 2u - v$. Clearly we have that w is already in the span of u and v. Hence w is redundant. On the other hand it turns out that u and v are linearly indpendent and span $\mathbb{R}^2$. So they will form a basis for $\mathbb{R}^2$.

Definition 2.5. Let V be a vector space and $\mathcal{B} \subset V$ be a subset, then $\mathcal{B}$ is a *basis* for V if

1. $\mathcal{B}$ spans V and
2. $\mathcal{B}$ is linearly independent.

Example 2.6. *The standard basis for $\mathbb{R}^n$:* Let e_i be a vector in $\mathbb{R}^n$ whose i-th entry is 1 and all other entries is 0. Then $\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$

For example if $n = 2$, then

$$\{e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

is a basis for $\mathbb{R}^2$. If $n = 3$, then

$$\{e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$$

is a basis for $\mathbb{R}^3$.

Example 2.7. *The standard basis for $\mathbb{P}(n)$:* The set

$$\{1, x, x^2, \dots, x^n\}$$

form a basis for $\mathbb{P}(n)$. What about $\mathbb{P}$? This case is a bit more complicated since $\mathbb{P}$ alows polynomials of arbitrarily large degree. A basis still exists but it will no longer be finite. In fact

$$\{1, x, x^2, x^3, x^4, \dots\}$$

forms a basis for $\mathbb{P}$ with an infinite number of elements.

The number of vectors in a basis is so important that we give it its own name, dimension.

Definition 2.8. Let V be a vector space $\mathcal{B} \subset V$ be a basis, then the dimension of V is the number of vectors in $\mathcal{B}$. If the number of vectors in $\mathcal{B}$ is finite we say that V is *finite dimensional*.

Example 2.9. The dimension of $\mathbb{R}^n$ is n since $\{e_1, \dots, e_n\}$ is a basis with n elements.

Example 2.10. The dimension of $\mathbb{P}(n)$ is $n+1$ since $\{1, x, x^2, \dots, x^n\}$ is a basis with $n+1$ elements. Since a basis for $\mathbb{P}$ has an infinite number of elements, then we say that $\mathbb{P}$ is infinite dimensional.

There is one major potentional issue with our definition of dimension. What if there exists two basis for a vector space with a different number of vectors? It turns out that linear indpendence (our minimality condition) forces all basis for a vector space to have the same number of elements.

Theorem 2.11. *Let V be a vector space and $\mathcal{B}$ and $\mathcal{D}$ be basis for V, then $\mathcal{B}$ and $\mathcal{D}$ have the same size as sets. In particular, if $\{v_1, \dots, v_n\}$ is a basis for V, then any other basis of V also has n elements.*

This theorem would not be true if we did not require our basis to be linearly independent.

3 Linear transformations

Linear transformations are the main focus of linear algebra everything we have learned thus far are just tools for studying linear transformations. A linear transformation is a special type of function between vector spaces $T : V \to W$ that respects the linear structure of V and W.

Definition 3.1. Let V and W be vector spaces and $T : V \to W$ a function. If for all u and v in V and real number c we have that

1. $T(u + v) = T(u) + T(v)$
2. and $T(c \cdot u) = c \cdot T(u)$,

then we say that T is a *linear transformation*.

Theorem 3.2. *Let V and W be vector spaces and let $\{v_1, \ldots, v_n\}$ be a basis for V with n elements, then any choice of n vectors $\{w_1, \ldots, w_n\}$ in W, possibly allowing duplicates, determines a unique linear transformation $T : V \to W$ such that $T(v_1) = w_1, \ldots, T(v_n) = w_n$. Furthermore, every linear transformation is determined this way.*

Example 3.3. Let us classify all linear transformations T from $\mathbb{R}^2$ to $\mathbb{R}^2$. We know that $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$, thus by the theorem any linear transformation is determined by where we send e_1 and e_2. Suppose that

$$T(e_1) = \begin{pmatrix} a \\ c \end{pmatrix}$$

and

$$T(e_2) = \begin{pmatrix} b \\ d \end{pmatrix}$$

where a, b, c, d are real numbers. We can use matrices to express this relation succinctly in a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Using matrix multiplication we see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

Hence the linear transformation T may be expressed as multiplication by the matrix A, i.e., $T(\vec{x}) = A\vec{x}$.

The set of all linear transformations from $\mathbb{R}^2$ to ^{2} is then the same as the set $\mathcal{M}(2 \times 2)$ consisting of all 2 by 2 matrices with real entries.

Problem 9. Describe the set of all linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$.

Example 3.4. The set of all linear transformations $T : \mathbb{R}^m \to \mathbb{R}^n$ is the set $\mathcal{M}(n \times m)$ consisting of all n by m matrices with real entries.

Problem 10. Show that $\mathcal{M}(2 \times 2)$ is a vector space.

Problem 11. Let V and W be vector spaces, then show that the set of all linear transformations between V and W is also a vector space.

Problem 12. Show that the derivative operator is a linear transformation from $\mathbb{P}(2)$ to $\mathbb{P}(1)$. In general the derivative operator is a linear transfromation from $\mathbb{P}(n)$ to $\mathbb{P}(n-1)$.

Problem 13. Show that the function $T : \mathbb{P}(3) \to \mathbb{R}$ defined by $T(f(x)) = \int_0^1 f(x)dx$ is a linear transformation.

Problem 14. Show that the functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x) = x + 1$ and $g(x) = x^2$ are not linear transfromations.

A special case of linear transformations is when $V = W$.

Definition 3.5. An endomorphism is a linear transfromation $T : V \to V$.

Since T maps into the same space one may ask what happens to a basis of V under this transfromation? In particular, if $\{v_1, \ldots, v_n\}$ is a basis for V, then is $\{T(v_1), \ldots, T(v_n)\}$ still a basis for V? The answer in general is no. When $\{T(v_1), \ldots, T(v_n)\}$ is a basis for V this is very special and gets its own name.

Definition 3.6. Let V be a vector space with basis $\{v_1, \ldots, v_n\}$ and let $T : V \to V$ be an endomorphism. T is called *invertible* if $\{T(v_1), \ldots, T(v_n)\}$ is also a basis for V.