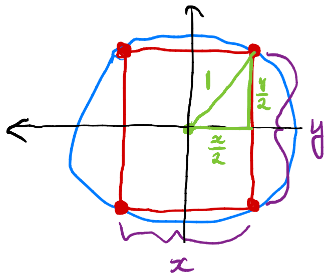
Lesson 10: Lagrange Multipliers

eg. What are the dimensions of a rectangle inscribed in a circle of radius 1 so that the area of the rectangle is maximized?



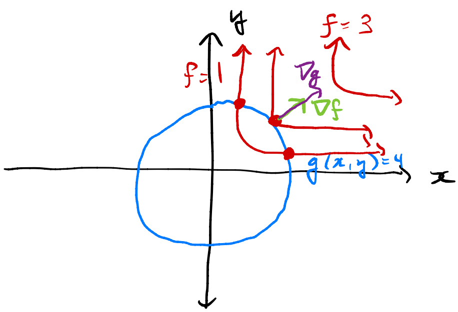
Area of □ : $f(x,y) = xy$

Constraint: $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

$g(x,y) = x^2 + y^2 = 4$

$x, y > 0$

Want to find abs max of $f(x,y)$ subject to the constraint $g(x,y) = 4$.



Want the level curve to be tangent to the constraint.

Recall: gradient vector is normal to the level curves

So...
$$\nabla f = \lambda \nabla g$$
$$g = 4$$

$f = xy \qquad g = x^2 + y^2$

$\nabla f = \langle y, x \rangle \qquad \nabla g = \langle 2x, 2y \rangle$

$$\langle y, x \rangle = \langle 2\lambda x, 2\lambda y \rangle$$

$$\boxed{\begin{aligned} y &= 2\lambda x \\ x &= 2\lambda y \\ x^2 + y^2 &= 4 \end{aligned} \qquad \text{Solve for } x, y \quad (x, y > 0)}$$

Idea: Solve for λ $\quad (x, y > 0)$

$$\frac{y}{2x} = \lambda = \frac{x}{2y}$$

$$2y^2 = 2x^2$$

$$y^2 = x^2$$

$$y = \pm x$$

$$y = x$$

$$x^2 + y^2 = 4$$

$$x^2 + x^2 = 4$$

$$2x^2 = 4$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

$$x = \sqrt{2} \qquad (x, y) = (\sqrt{2}, \sqrt{2})$$

The rectangle has dim. $\sqrt{2}$ by $\sqrt{2}$.

Method of Lagrange Multipliers

Optimize an objective function f subject to some constraint $g = k$.

Solve the system:

$$\nabla f = \lambda \nabla g$$
$$g = k$$

eg. Find the abs. max. of $f(x,y) = 26x^{3/2}y$ subject to the const. $y = 56 - \frac{x}{2}$.

$$g(x,y) = y + \frac{1}{2}x$$

$$g = 56$$
$$y + \frac{1}{2}x = 56$$
$$y = 56 - \frac{1}{2}x$$

$$\nabla f = \langle 39x^{1/2}y, 26x^{3/2} \rangle$$

$$\nabla g = \langle \frac{1}{2}, 1 \rangle$$

$$\nabla f = \lambda \nabla g$$
$$g = 56$$

$$\Rightarrow$$

$$39x^{1/2}y = \frac{1}{2}\lambda$$
$$26x^{3/2} = \lambda$$
$$y + \frac{1}{2}x = 56$$

Solve for λ:

$$78x^{1/2}y = \lambda = 26x^{3/2}$$

$$78x^{1/2}y - 26x^{3/2} = 0$$
$$26x^{1/2}(3y - x) = 0$$

$$26x^{1/2} = 0 \quad \text{or} \quad 3y - x = 0$$
$$x = 0 \qquad\qquad x = 3y$$

Case $x = 0$:

$$y + \frac{1}{2}x = 56$$
$$y = 56$$

$(0, 56)$

Case $x = 3y$:

$$y + \frac{1}{2}x = 56$$
$$y + \frac{3}{2}y = 56$$
$$y = 22.4$$

$$x = 3y = 67.2$$

$(67.2, 22.4)$

Lastly :

$f(0, 56) = 0$ ← abs min

$f(67.2, 22.4) \approx \underline{320830.225}$ ← abs max eg. Find the abs min and max of $f(x,y) = 4 - x^2 - y^2$ subject to the const. $g(x,y) = 4x^2 + y^2 - 4 = 0$.

$$\nabla f = \langle -2x, -2y \rangle$$
$$\nabla g = \langle 8x, 2y \rangle$$

$$\boxed{\begin{array}{l} \nabla f = \lambda \nabla g \\ g = 0 \end{array} \Rightarrow \begin{array}{l} -2x = 8\lambda x \\ -2y = 2\lambda y \\ 4x^2 + y^2 - 4 = 0 \end{array}}$$

$$8\lambda x + 2x = 0$$
$$2x(4\lambda + 1) = 0$$

$2x = 0$ or $4\lambda + 1 = 0$

$x = 0$ $\qquad \lambda = -1/4$

Case $x = 0$:

$$4x^2 + y^2 - 4 = 0$$
$$y^2 = 4$$
$$y = \pm 2$$

$(0,-2)$ and $(0,2)$

Case $\lambda = -1/4$: Examine $-2y = 2\lambda y$

$$-2y = -\tfrac{1}{2} y$$
$$4y = y$$
$$3y = 0$$
$$y = 0$$

$$4x^2 + y^2 - 4 = 0$$
$$4x^2 = 4$$
$$x^2 = 1$$
$$x = \pm 1$$

$(-1, 0)$ and $(1, 0)$.

$$\left.\begin{aligned} f(0,-2) &= 0 \\ f(0,2) &= 0 \end{aligned}\right\} \text{abs min}$$

$$\left.\begin{aligned} f(-1,0) &= 3 \\ f(1,0) &= 3 \end{aligned}\right\} \text{abs max.}$$

eg. Find abs min and max of $f(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = x^2+y^2+z^2 = 1$.

$\nabla f = \lambda \nabla g$
$g = 1$

$$\boxed{\begin{aligned} yz &= \lambda(2x) \\ xz &= \lambda(2y) \\ xy &= \lambda(2z) \\ x^2+y^2+z^2 &= 1 \end{aligned}}$$

Idea: Assume $x, y, z \neq 0$
solve for λ.

$$\lambda = \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

$2y^2z = 2x^2z$

$y^2 = x^2$
$y = \pm x$

$2xz^2 = 2xy^2$

$z^2 = y^2$
$z = \pm y$

So... $x^2 = y^2 = z^2$

$x^2+y^2+z^2 = 1$
$3x^2 = 1$
$x = \pm 1/\sqrt{3}$

So... $x = \pm 1/\sqrt{3}$ $y = \pm 1/\sqrt{3}$ $z = \pm 1/\sqrt{3}$

What if at least one of x, y or z is 0?

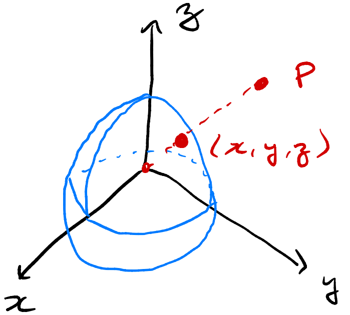
$$f(x, y, z) = xyz = 0$$

x	y	z	$f(x, y, z)$
$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{27}$
$1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{27}$
$1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{27}$
$1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{27}$
$-1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{27}$
$-1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{27}$
$-1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{27}$
$-1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{27}$
one of x, y or z is 0			0

abs min of $-1/\sqrt{27}$

abs max of $1/\sqrt{27}$

eg. Find point on the sphere $x^2+y^2+z^2=4$ that is closest to $P(6,8,8)$.



$x, y, z > 0$

Distance between (x,y,z) and P

$$\sqrt{(x-6)^2+(y-8)^2+(z-8)^2}$$

Idea: instead of minimizing distance minimized distance squared.

$$f(x,y,z) = (x-6)^2+(y-8)^2+(z-8)^2$$
$$g(x,y,z) = x^2+y^2+z^2 = 4$$

$$(x, y, z > 0)$$

$$\boxed{\begin{matrix} \nabla f = \lambda \nabla g \\ g = 4 \end{matrix} \Rightarrow \begin{matrix} 2(x-6) = \lambda(2x) \\ 2(y-8) = \lambda(2y) \\ 2(z-8) = \lambda(2z) \\ x^2+y^2+z^2 = 4 \end{matrix}}$$

$$\lambda = \underbrace{\frac{x-6}{x} = \frac{y-8}{y}}\underbrace{= \frac{z-8}{z}}$$

$$y(x-6) = x(y-8) \qquad z(y-8) = y(z-8)$$

$$yx - 6y = xy - 8x \qquad yz - 8z = yz - 8y$$

$$-6y = -8x \qquad -8z = -8y$$

$$y = 4/3\ x \qquad z = y$$

$$x^2 + y^2 + z^2 = 4$$

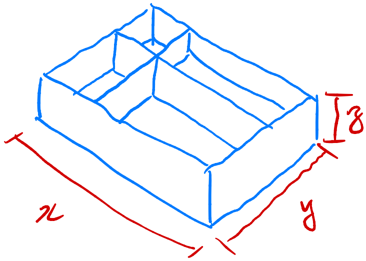
$$x^2 + \frac{16}{9}x^2 + \frac{16}{9}x^2 = 4$$

$$x^2 = \frac{36}{41}$$

$$x = \frac{6}{\sqrt{41}} \qquad (x > 0)$$

$\left(6/\sqrt{41},\ 8/\sqrt{41},\ 8/\sqrt{41} \right)$ this point is on the sphere and closest to P.

eg. We have an open top cardboard box w/ partitions



Find dim. of the box that max. volume and uses 243 in^2 of cardboard.

Objective : $f(x,y,z) = xyz$

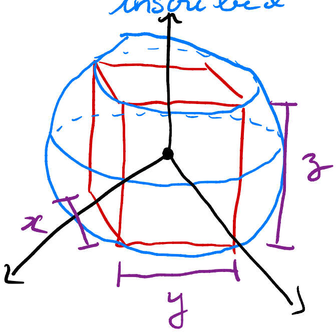
Constraint : $g(x,y,z) = xy + 3xz + 3yz = 243$

$$(x, y, z > 0)$$

$\nabla f = \lambda \nabla g$

$g = 243$

eg. Find the dim. of a rectangular box with max volume that can be inscribed in the unit sphere (radius 1).



Volume : $f(x,y,z) = xyz$

Constraint : $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$

$g(x,y,z) = x^2 + y^2 + z = 4$