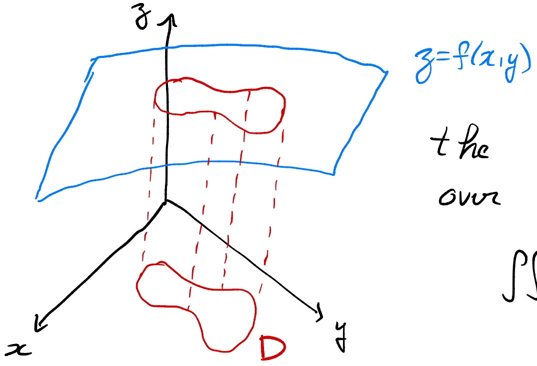
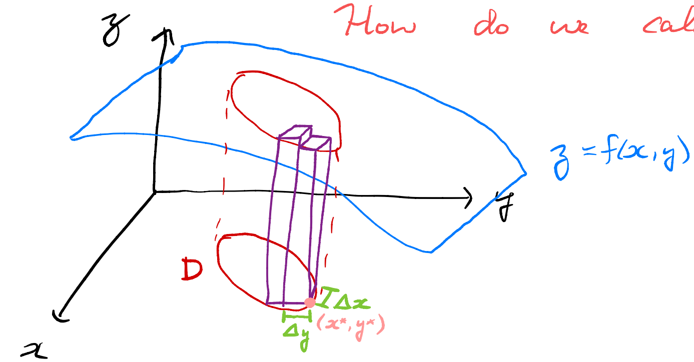
Lesson 11: Double integrals over rectangular regions



the volume under $f(x,y)$ over a region D is

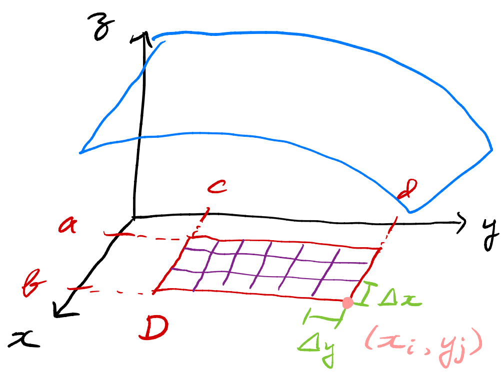
$$\iint_D f(x,y)\,dA$$

How do we calculate $\iint_D f(x,y)\,dA$?



Volume of one box: $f(x^*,y^*)\Delta x \Delta y$

$$\iint_D f(x,y)\,dA \approx \sum f(x^*,y^*)\Delta x \Delta y$$



$$\iint_D f(x,y)\,dA \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_j)\Delta x \Delta y$$

$$\iint_D f(x,y)\,dA = \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_j)\Delta x \Delta y$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$\underbrace{\int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{\substack{\text{integrate wrt } x \\ \text{treat } y \text{ as a constant}}}\,dy}_{\text{integrate wrt } y}$$

eg Evaluate $\int_1^2 \int_0^3 (x+4y)\,dy\,dx$

$$= \int_1^2 \left(xy + 2y^2\right)\Big|_{y=0}^{y=3}\,dx$$

$$= \int_1^2 \left(3x + 2\cdot 9 - (0+0)\right)dx$$

$$= \int_1^2 (3x+18)\,dx$$

$$= 45/2$$

What about $\int_0^3 \int_1^2 (x+4y)\,dx\,dy$?

Special property of rectangular regions

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

eg. Evaluate $\int_1^3 \underbrace{\int_{3\pi/2}^{2\pi} x \sin y \, dy}_{\substack{\text{integrate wrt } y \\ \text{treat } x \text{ as a constant}}} dx$

$$\int_{3\pi/2}^{2\pi} x \sin y \, dy = -x \cos y \Big|_{y=3\pi/2}^{y=2\pi}$$

$$= -x \left(\cos(2\pi) - \cos\left(\tfrac{3\pi}{2}\right) \right)$$

$$= -x (1 - 0)$$

$$= -x$$

$$\int_1^3 -x \, dx = -\frac{1}{2} x^2 \Big|_1^3 = -4$$

$$= \int_0^2 \left(x^3 \left(\int_0^1 y^3 e^{x^2y^4}\, dy \right) dx \right)$$

eg. Evaluate $\int_0^2 \int_0^1 x^3y^3e^{x^2y^4}\, dy\, dx$

$$\int_0^1 x^3y^3e^{x^2y^4}\, dy = \int_{y=0}^{y=1} \frac{1}{4} x e^u\, du$$

$u = x^2y^4$
$du = 4x^2y^3\, dy$

$$= \frac{1}{4} x e^u \Big|_{y=0}^{y=1}$$

$$= \frac{1}{4} x e^{x^2y^4} \Big|_{y=0}^{y=1}$$

$$= \frac{1}{4} x \left(e^{x^2} - 1 \right)$$

$$= \frac{1}{4} \left(x e^{x^2} - x \right)$$

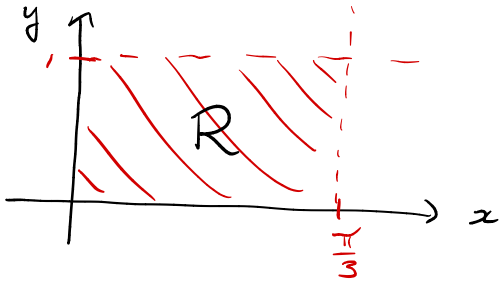
$$\int_0^2 \frac{1}{4} \left(x e^{x^2} - x \right) dx$$

$$= \frac{1}{4} \left(\frac{1}{2} e^{x^2} - \frac{1}{2} x^2 \right) \Big|_0^2$$

$$= \frac{1}{4} \left(\frac{1}{2} e^4 - \frac{1}{2} (2)^2 - \left(\frac{1}{2} - 0 \right) \right)$$

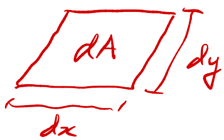
$$= \frac{e^4 - 5}{8}$$

eg. Evaluate $\iint_R x^2\cos(xy)\,dA$ where $R = \{(x,y) \mid 0 \le x \le \pi/3,\ 0 \le y \le 1\}$.



$dx \approx$ small (infitimal) length in the x-axis
$dy \approx$ " " y-axis

$dA \approx$



$$dA = dx\,dy = dy\,dx$$

small (infinit—) area.

$$\iint_R x^2\cos(xy)\,dA = \int_0^{\pi/3}\int_0^1 x^2\cos(xy)\,dy\,dx$$

$$\int_0^1 x^2\cos(xy)\,dy = x^2\int_0^1 \cos(xy)\,dy$$

$u = xy$
$du = x\,dy$

$$= x\sin(xy)\Big|_{y=0}^{y=1}$$

$$= x(\sin x - 0)$$
$$= x\sin x$$

$$\int_0^{\pi/3} x \sin x \, dx = -x\cos x \Big|_0^{\pi/3} + \int_0^{\pi/3} \cos x \, dx$$

$u = x \qquad du = dx$

$dv = \sin x \, dx \qquad v = -\cos x$

$$= -\frac{\pi}{3}\cos\left(\frac{\pi}{3}\right) - 0 + \sin x \Big|_0^{\pi/3}$$

$$= -\frac{\pi}{3}\,\frac{1}{2} + \frac{\sqrt{3}}{2} - 0$$

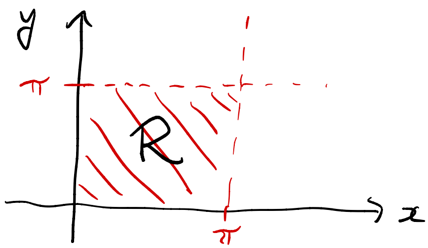
$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6} \quad \checkmark$$

eg. Compute the average value of $f(x,y) = \sin(2x+y)$ over the region $R = \{(x,y) \mid 0 \le x \le \pi,\ 0 \le y \le \pi\}$

To take the average of a list of numbers

$$\text{average} = \frac{\Sigma \text{ of all numbers in list}}{\#\text{ items in the list.}}$$

$$\text{average} = \frac{\iint_R f(x,y)\,dA}{\text{area}(R)}$$



$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \sin(2x+y)\,dy\,dx$$

$$= \frac{1}{\pi^2}\int_0^{\pi} -\cos(2x+y)\Big|_{y=0}^{y=\pi}\,dx$$

$$= \frac{-1}{\pi^2}\int_0^{\pi} \cos(2x+\pi) - \cos(2x)\,dx$$

$$= \frac{-1}{\pi^2}\left(\frac{1}{2}\sin(2x+\pi) - \frac{1}{2}\sin(2x)\right)\Big|_0^{\pi}$$

$$= \frac{-1}{\pi^2}\left(\frac{1}{2}\underbrace{\sin(3\pi)}_{0} - \frac{1}{2}\underbrace{\sin(2\pi)}_{0} - \left(\frac{1}{2}\underbrace{\sin(\pi)}_{0} - \frac{1}{2}\underbrace{\sin(0)}_{0}\right)\right)$$

$$= 0$$