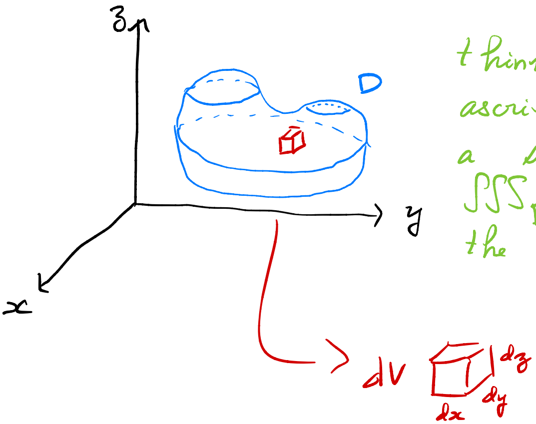
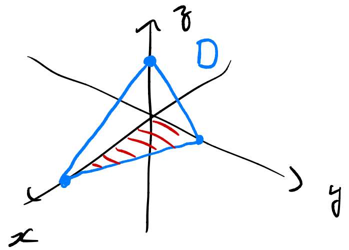
Lesson 14: Triple Integrals



think of $f(x,y,z)$ as ascribing to every point (x,y,z) a specific density, so $\iiint_D f(x,y,z)\,dV$ represent the weight of D.

$$dV = dx\,dy\,dz = dy\,dx\,dz = dz\,dx\,dy$$
$$= dx\,dz\,dy = dy\,dz\,dx = dz\,dy\,dx$$

eg. Use a triple integral to find the volume of the solid under $x + 2y + 2z = 4$ in the first octant.



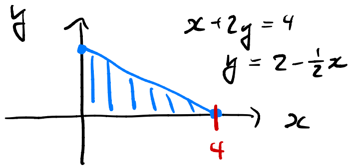
$$z = 2 - \frac{1}{2}x - y$$

x inter: $x = 4$ $(4, 0, 0)$

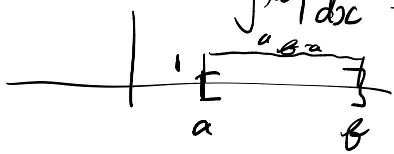
y inter: $2y = 4$ $(0, 2, 0)$

z inter: $2z = 4$ $(0, 0, 2)$

As double integral: $V = \int_0^4 \int_0^{2-\frac{1}{2}x} 2 - \frac{1}{2}x - y \; dy\,dx$



$$\int_a^b 1\,dx = b - a$$



As a triple integral: $V = \iiint_D 1 \; dV$

$$V = \int_0^4 \int_0^{2-\frac{1}{2}x} \int_0^{2-\frac{1}{2}x-y} 1 \; dz\,dy\,dx$$

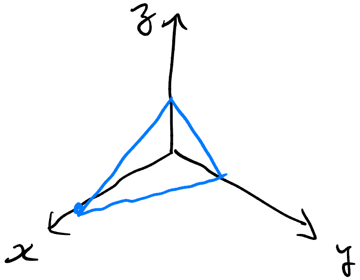
$$= \int_0^4 \int_0^{2-\frac{1}{2}x} \left(2 - \frac{1}{2}x - y\right) dy\,dx$$

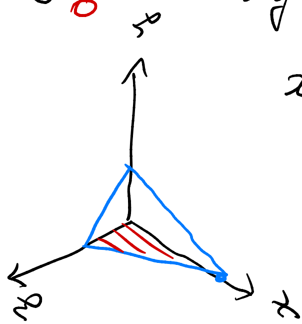
$$= 8/3$$

eg. In the previous problem we used $dV = dz\,dy\,dx$

What if $dV = dy\,dx\,dz$?

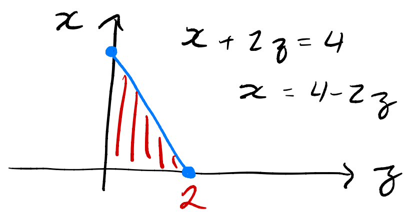
$$V = \int_0^{2} \int_0^{4-2z} \int_0^{2-\frac{1}{2}x-z} 1 \; dy\,dx\,dz$$





$x + 2y + 2z = 4$

$y = 2 - \frac{1}{2}x - z$



$x + 2z = 4$

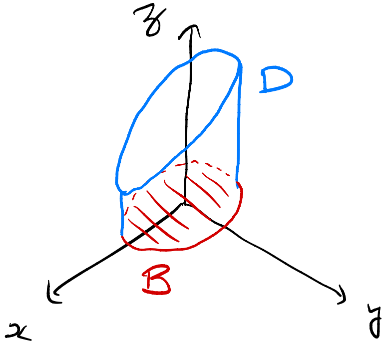
$x = 4 - 2z$

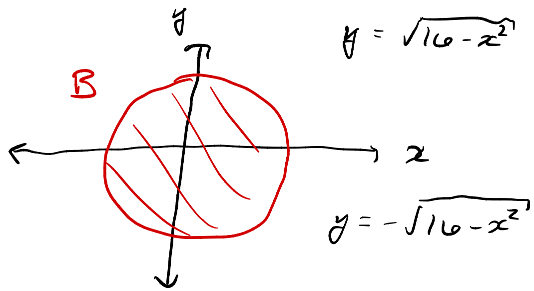
$$V = \int_0^{2} \int_0^{4-2z} y \Big|_0^{2-\frac{1}{2}x-z} dx\,dz$$

$$= \int_0^{2} \int_0^{4-2z} \left(2 - \frac{1}{2}x - z\right) dx\,dz$$

$$= 8/3$$

eg. Use a triple integral to find the volume inside cylinder $x^2+y^2=16$ below the plane $z=5-x$ and above $z=0$.





$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$V = \iiint_D 1\,dV = \iint_B \int_0^{5-x} dz\,dA$$

$$= \iint_B 5 - x\,dA$$

$$= \int_0^{2\pi} \int_0^4 (5 - r\cos\theta)\, r\,dr\,d\theta$$

$$= 80\pi$$

Good trig ids to know: (not exhaustive)

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

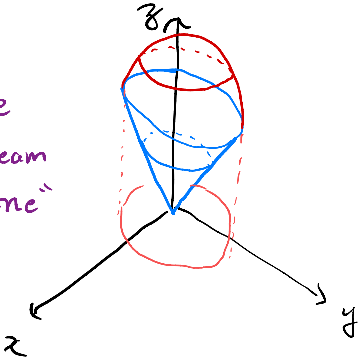
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

eg. Use a triple integral to find the volume of the solid bounded below by $z=\sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2=200$.

"ice cream cone"



let us preform a change of coordinates to "cylindrical coordinates" (r, θ, z)

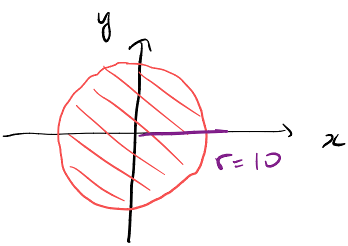
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$\begin{aligned} dV &= dz\, dA \\ &= dz\,(r\,dr\,d\theta) \\ &= r\,dz\,dr\,d\theta \end{aligned}$$

$$\begin{aligned} x^2+y^2+z^2 &= 200 \\ r^2+z^2 &= 200 \\ z &= \pm\sqrt{200-r^2} \end{aligned}$$

$$\begin{aligned} z &= \sqrt{x^2+y^2} \\ &= \sqrt{r^2} \\ &= r \end{aligned}$$

$$V = \int_0^{2\pi}\int_0^{10}\int_r^{\sqrt{200-r^2}} \boxed{1}\, r\,dz\,dr\,d\theta$$



$$\begin{aligned} \sqrt{200-r^2} &= z = r \\ 200-r^2 &= r^2 \\ 2r^2 &= 200 \\ r^2 &= 100 \\ r &= 10 \end{aligned}$$

$$= \int_0^{2\pi}\int_0^{10} r z \Big|_{z=r}^{z=\sqrt{200-r^2}} dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{10} r\left(\sqrt{200-r^2} - r\right) dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{10} r\sqrt{200-r^2}\, dr\, d\theta - \int_0^{2\pi} \int_0^{10} r^2\, dr\, d\theta$$

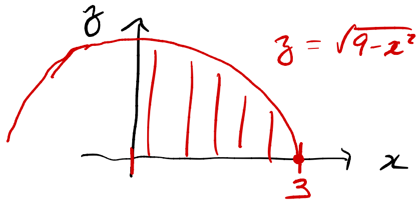
$u = 200 - r^2$

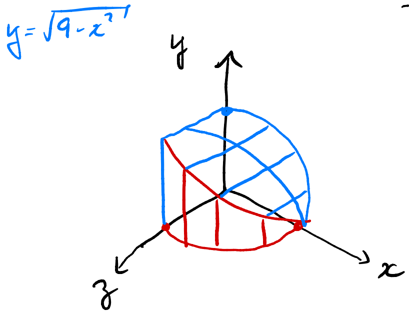
$$= \frac{4000(\sqrt{2}-1)\pi}{3}$$

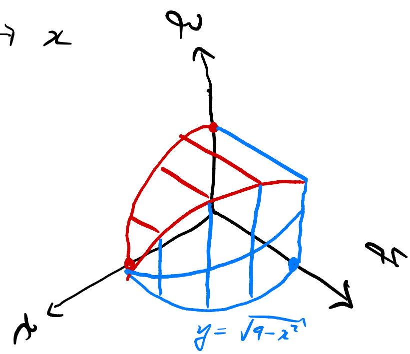
eg Rewrite $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dy\,dz\,dx$ as a $dz\,dy\,dx$ integral.

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} 1 \; dz\,dy\,dx$$

xz - plane :

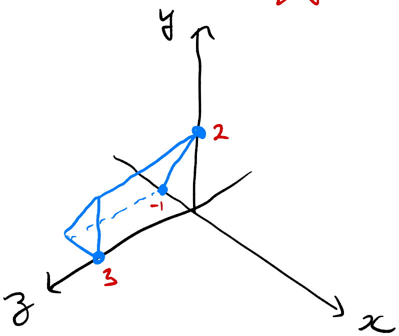


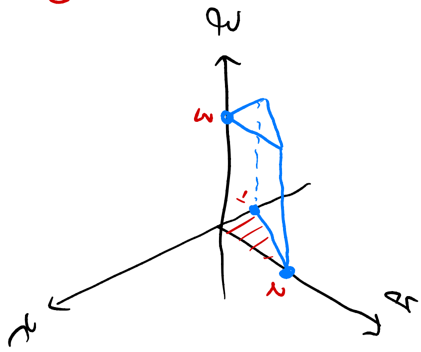




eg. Rewrite $\int_0^3 \int_{-1}^0 \int_0^{2x+2} dy\, dx\, dz$
as $dz\, dx\, dy$.

$$\int_0^2 \int_{\frac{1}{2}y-1}^{0} \int_0^3 1 \; dz\, dx\, dy$$





$y = 2x + 2$
$x = \frac{1}{2} y - 1$