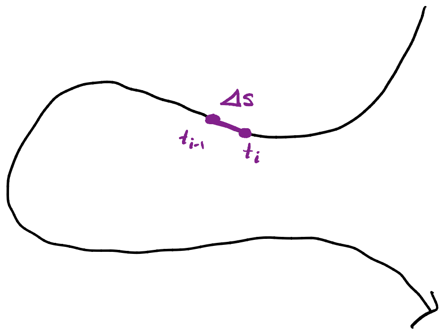
Lesson 17: Line integrals.

We can integrate on an interval w/ dx
surface w/ dA
solid w/ dV



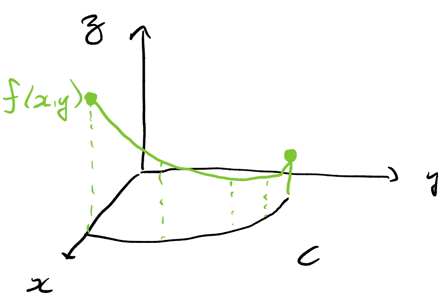
$C: \quad \vec{r}(t) = \langle x(t), y(t) \rangle$
$a \le t \le b$

length of a small segment of $C = \Delta s$
$= \int_{t_{i-1}}^{t_i} |\vec{r}\,'(t)|\, dt$

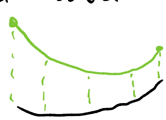
As segment gets smaller $\Delta s \to ds = |\vec{r}\,'(t)|\, dt$

The integral of $f(x,y)$ along C

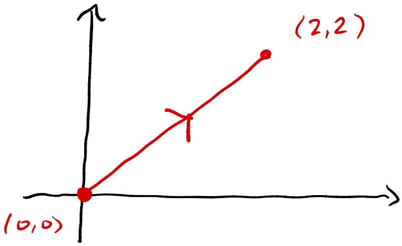
$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t))\, |\vec{r}\,'(t)|\, dt$$



$\int_C f(x,y)\, ds =$ signed area of



eg. Evaluate $\int_C x^2 + e^y \, ds$ where C is the line segment from $(0,0)$ to $(2,2)$.



$$\vec{r}(t) = \langle 0, 0 \rangle + t \langle 2, 2 \rangle$$
$$= \langle 2t, 2t \rangle$$
$$0 \leq t \leq 1$$

$$\int_C x^2 + e^y \, ds = \int_0^1 \left((2t)^2 + e^{2t} \right) \sqrt{(2)^2 + (2)^2} \, dt$$
$$= 2\sqrt{2} \int_0^1 (4t^2 + e^{2t}) \, dt$$
$$= \sqrt{2} \left(8/3 + e^2 - 1 \right)$$

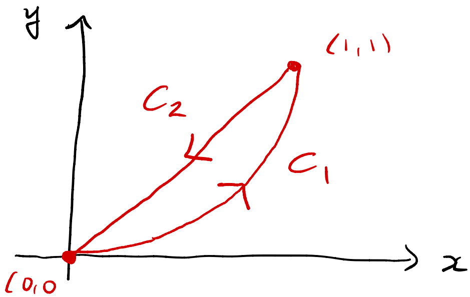
What if we use a different $\vec{r}(t)$?

$$\vec{r}(t) = \langle t, t \rangle \qquad 0 \leq t \leq 2$$

$$\int_C x^2 + e^y \, ds = \int_0^2 (t^2 + e^t) \sqrt{(1)^2 + (1)^2} \, dt$$
$$= \sqrt{2} \left(8/3 + e^2 - 1 \right)$$

Answer: no!

eg. Evaluate $\int_C x + \sqrt{y}\, ds$ where C goes from $(0,0)$ to $(1,1)$ along $y = x^2$ and from $(1,1)$ to $(0,0)$ along $y = x$.



$C_1: \vec{r}_1(t) = \langle t, t^2 \rangle \quad 0 \le t \le 1$

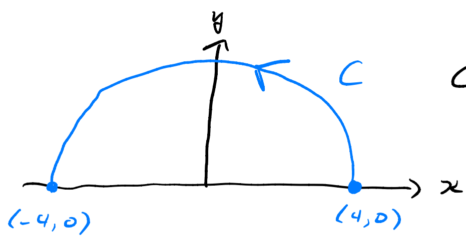
$$C_2: \vec{r}_2(t) = \langle 1,1 \rangle + t\langle 0-1, 0-1 \rangle = \langle 1-t, 1-t \rangle \quad 0 \le t \le 1$$

$$\int_C x + \sqrt{y}\, ds = \int_{C_1} x + \sqrt{y}\, ds + \int_{C_2} x + \sqrt{y}\, ds$$

$$= \int_0^1 \left(t + \sqrt{t^2}\right)\sqrt{(1)^2 + (2t)^2}\, dt + \int_0^1 \left(1 - t + \sqrt{1-t}\right)\sqrt{(-1)^2 + (-1)^2}\, dt$$

$$= \frac{1}{6}\left(5\sqrt{5} - 1\right) + \frac{7}{3}\sqrt{2}$$

eg. Evaluate $\int_C x + y\, ds$ where C is the half circle in the upper half plane. from $(4,0)$ to $(-4,0)$ orientated counter clock wise.

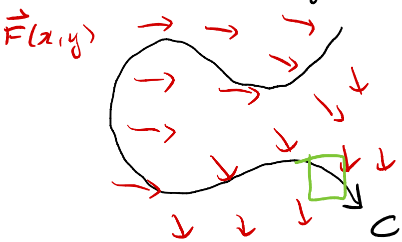


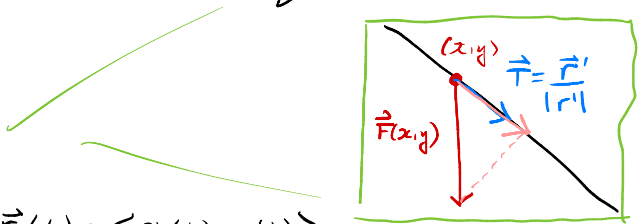
$C: \vec{r}(t) = \langle 4\cos t, 4\sin t \rangle$
$0 \le t \le \pi$

$$\int_C x + y\, ds = \int_0^{\pi} \left(4\cos t + 4\sin t\right)\sqrt{(-4\sin t)^2 + (4\cos t)^2}\, dt$$

$$= 4 \cdot 4 \int_0^{\pi} \cos t + \sin t \, dt = 32$$

Line integrals over vector fields





$$C : \vec{r}(t) = \langle x(t), y(t) \rangle$$
$$a \le t \le b$$

$$\left| \text{proj}_{\vec{T}} \vec{F}(x,y) \right| = \frac{\vec{F}(x,y) \cdot \vec{T}}{|\vec{T}|} = \vec{F}(x,y) \cdot \vec{T}$$

$\int_C \vec{F}(x,y) \cdot \vec{T} \, ds$ = accumulation of the flow which is parallel to C.

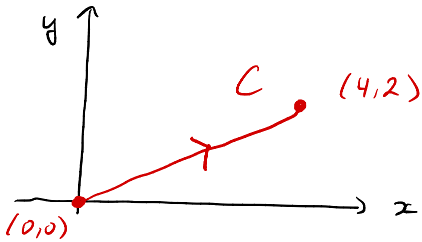
$$\int_C \vec{F}(x,y) \cdot \vec{T} \, ds = \int_a^b \vec{F}(x(t), y(t)) \cdot \frac{r'(t)}{|r'(t)|} \cdot |r'(t)| \, dt$$
$$= \int_a^b \vec{F}(x(t), y(t)) \cdot r'(t) \, dt$$
$$= \int_C \vec{F} \cdot d\vec{r}$$

Recall: Work = (Force)(displacement)

View $\vec{F}$ as describing a force acting on an object.

$\int_C \vec{F} \cdot \vec{T} \, ds$ = Work done to an object moving through $\vec{F}$ along C.

eg. Evaluate $\int_C \vec{F} \cdot \vec{T} \, ds$ where $\vec{F}(x,y) = \langle y, -2x \rangle$, C is the line segment from $(0,0)$ to $(4,2)$.

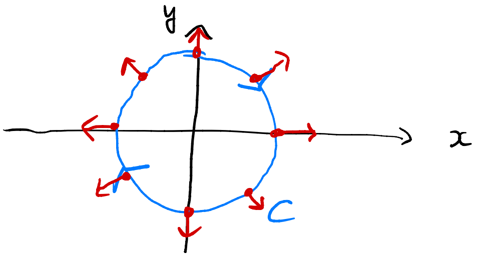


$$C: \vec{r}(t) = \langle 0,0 \rangle + t\langle 4,2 \rangle$$
$$= \langle 4t, 2t \rangle$$
$$0 \leq t \leq 1$$

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_0^1 \langle 2t, -2(4t) \rangle \cdot \langle 4, 2 \rangle \, dt$$

$$= \int_0^1 8t - 16t \, dt = -4$$

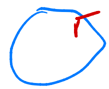
eg. Calculate the circulation of $\vec{F} = \langle x, y \rangle$ on the unit circle orientated clockwise.



circulation = flow parallel to a closed oriented curve

$$= \int_C \vec{F} \cdot \vec{T} \, ds$$

$C : \cancel{\vec{r}(t) = \langle \cos t , \sin t \rangle}$



Reverse time!

$$C : \vec{r}(t) = \langle \cos(-t), \sin(-t) \rangle$$
$$= \langle \cos t , -\sin(t) \rangle$$
$$0 \leq t \leq 2\pi$$

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_0^{2\pi} \langle \cos t, -\sin t \rangle \cdot \langle -\sin t, -\cos t \rangle \, dt$$
$$= \int_0^{2\pi} 0 \, dt = 0$$

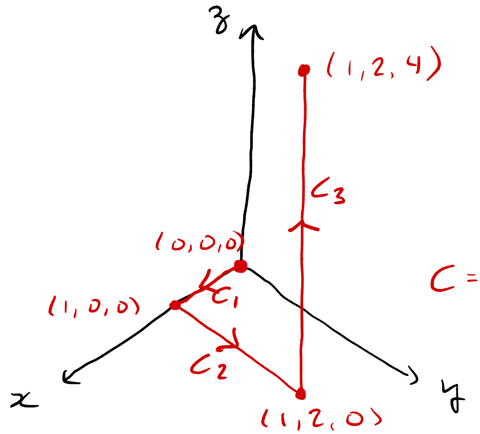
$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$$
$$C : \vec{r}(t) = \langle x(t), y(t) \rangle \qquad a \leq t \leq b$$

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_a^b \vec{F} \cdot \vec{r}' \, dt$$

$$= \int_a^b \langle P, Q \rangle \cdot \langle x', y' \rangle \, dt$$
$$= \int_a^b P x' \, dt + \int_a^b Q y' \, dt$$
$$= \int_C P \, dx + \int_C Q \, dy = \int_C P\,dx + Q\,dy$$

$$\boxed{\begin{aligned} dx &= x' \, dt \\ dy &= y' \, dt \end{aligned}}$$

eg. Let $\vec{F}(x,y,z) = \langle x, y, z\rangle$ be a force field and C is



An object is moving along C. Calculate work done on the object parallel to its movement.

$C = C_1 + C_2 + C_3$

$C_1 : \vec{r}(t) = \langle t, 0, 0\rangle \quad 0 \le t \le 1$

$C_2 : \vec{r}(t) = \langle 1, 2t, 0\rangle \quad 0 \le t \le 1$

$C_3 : \vec{r}(t) = \langle 1, 2, 4t\rangle \quad 0 \le t \le 1$

$$\text{Work} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_{C_1} + \int_{C2} + \int_{C_3}$$

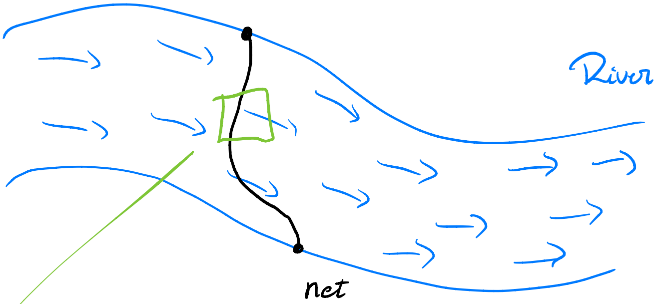
$$= \int_0^1 \langle t, 0, 0\rangle\cdot\langle 1, 0, 0\rangle\,dt$$

$$+ \int_0^1 \langle 1, 2t, 0\rangle\cdot\langle 0, 2, 0\rangle\,dt$$

$$+ \int_0^1 \langle 1, 2, 4t\rangle\cdot\langle 0, 0, 4\rangle\,dt$$

$$= \int_0^1 t\,dt + \int_0^1 4t\,dt + \int_0^1 16t\,dt$$

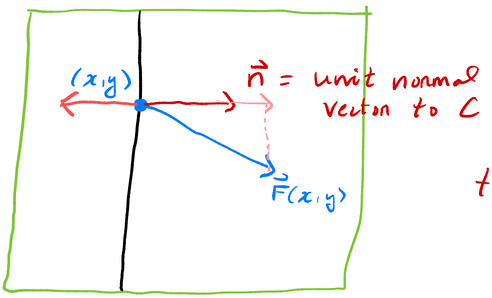
$$= 21/2$$



How much water passes through the net?

$\vec{F}(x,y)$ describe the flow of the water

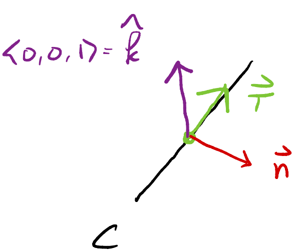
C be the curve representing the net.



$$= \int_C \vec{F} \cdot \vec{n}\, ds$$

this is called the flux

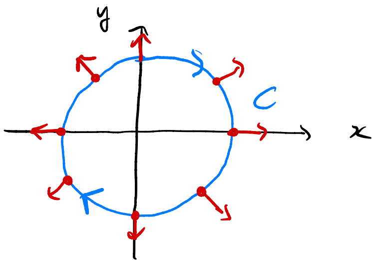
How do we calculate $\vec{n}$?



Convention:

$$\vec{n} = \vec{T} \times \hat{k} = \vec{T} \times \langle 0, 0, 1\rangle$$

eg. Calcute the flux of $\vec{F} = \langle x, y \rangle$ along the unit circle orientated clock wise



$C: \; r(t) = \langle \overset{x}{\cos t}, \overset{y}{-\sin t} \rangle$

$0 \leq t \leq 2\pi$

$r'(t) = \langle -\sin t, -\cos t \rangle$

$|r'(t)| = 1$

$\vec{T} = \langle -\sin t, -\cos t \rangle$

$$\text{flux} = \int_C \vec{F} \cdot \vec{n} \, ds$$

$$\vec{n} = \langle -\sin t, -\cos t, 0 \rangle \times \langle 0, 0, 1 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & -\cos t & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle -\cos t - 0, -(-\sin t - 0), 0 - 0 \rangle$$

$$= \langle -\cos t, \sin t, 0 \rangle$$

$$\vec{n} = \langle -\cos t, \sin t \rangle$$

$ds = |r'(t)| \, dt$

$$\int_C F \cdot \vec{n} \, ds = \int_0^{2\pi} \langle \cos t, -\sin t \rangle \cdot \langle -\cos t, \sin t \rangle \, (1) \, dt$$

$$= \int_0^{2\pi} -\cos^2 t - \sin^2 t \, dt$$

$$= -\int_0^{2\pi} dt = -2\pi$$