Lesson 18: Fundamental theorem of line integrals

Thm (Fundamental Thm of Calculus)

$$\int_a^b \frac{d}{dx}[f(x)]\,dx = f(b) - f(a)$$

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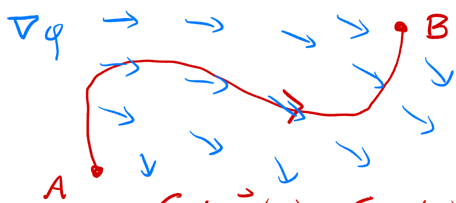

$C: \vec{r}(t) = \langle x(t), y(t) \rangle \qquad a \leq t \leq b$

$A = \vec{r}(a) \qquad B = \vec{r}(b)$

$$\int_C \nabla\varphi \cdot d\vec{r} = \int_a^b \nabla\varphi \cdot \vec{r}\,'\,dt$$

$$= \int_a^b \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$

$$= \int_a^b \left(\frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\cdot\frac{dy}{dt} \right) dt$$

$$= \int_a^b \frac{d\varphi}{dt}\,dt$$

$$= \int_a^b \frac{d}{dt}[\varphi]\,dt$$

$$= \varphi(\vec{r}(b)) - \varphi(\vec{r}(a))$$

$$\boxed{\int_C \nabla\varphi \cdot d\vec{r} = \varphi(B) - \varphi(A)}$$

We can choose any path from $A \to B$ and int doesn't change.

eg. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle x+y, x\rangle$ and C is the curve from $(0,0)$ to $(2,4)$ along $y = x^2$.

Find φ so that $\nabla\varphi = \vec{F}$

$$\langle \varphi_x, \varphi_y\rangle = \langle x+y, x\rangle$$

$\varphi_x = x + y$ $\qquad\qquad$ $\varphi_y = x$

antideriv. wrt. x

$\varphi = \frac{1}{2}x^2 + xy + h(y)$ $\qquad\qquad$ $\varphi = xy + g(x)$

$$\varphi = \frac{1}{2}x^2 + xy + c$$

choose $c = 0$, $\varphi = \frac{1}{2}x^2 + xy$

$$\nabla\varphi = \langle x+y, x\rangle = \vec{F}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\varphi\cdot d\vec{r} = \varphi(2,4) - \varphi(0,0)$$

$$= \left(\frac{1}{2}(2)^2 + (2)(4)\right) - \left(\frac{1}{2}(0)^2 + (0)(0)\right)$$

$$= 10$$

We say a vector field $\vec{F}$ is conservative if there is a function φ so $\nabla\varphi = \vec{F}$.

φ is called a potential function.

Fact: $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ is Conservative when

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

eg. Are the following conservative?

a) $\vec{F} = \langle x, y\rangle$ $\qquad 0 = 0$ ✓ $\quad \varphi = \frac{1}{2}x^2 + \frac{1}{2}y^2$

b) $\vec{F} = \langle y^2, x^2\rangle$ $\qquad 2y \neq 2x$ ✗

c) $\vec{F} = \langle x+y, x\rangle$ $\qquad 1 = 1$ ✓

only (a) and (c) are conservative

$$\nabla\varphi = \Big\langle \underset{\displaystyle P}{\frac{\partial\varphi}{\partial x}}, \underset{\displaystyle Q}{\frac{\partial\varphi}{\partial y}} \Big\rangle$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2\varphi}{\partial y\,\partial x} = \frac{\partial^2\varphi}{\partial y\,\partial x} = \frac{\partial Q}{\partial x}$$

$\vec{F} = \langle P, Q, R\rangle$ conservative if

$$P_y = Q_x, \quad P_z = R_x \quad \text{and} \quad Q_z = R_y$$

eg. Determine if

$$\vec{F} = \langle \overset{P}{2x\cos y - 2z^3},\ \overset{Q}{3 + 2ye^z - x^2\sin y},\ \overset{R}{y^2e^z - 6xz^2} \rangle$$

is conservative. If so find potential.

$P_y = Q_x \qquad -2x\sin y = -2x\sin y \quad ✓$

$P_z = R_x \qquad -6z^2 = -6z^2 \quad ✓$

$Q_z = R_y \qquad 2ye^z = 2ye^z \quad ✓$

$\vec{F}$ is conservative!

$\langle \varphi_x, \varphi_y, \varphi_z \rangle = \nabla \varphi(x, y, z) = \vec{F}$

$$\varphi_x = 2x\cos y - 2z^3 \xrightarrow[x]{\text{int wrt}} \varphi = x^2\cos y - 2xz^3 + f(y, z)$$

$$\varphi_y = 3 + 2ye^z - x^2\sin y \xrightarrow[y]{\text{int wrt}} \varphi = 3y + y^2e^z + x^2\cos y + h(x, z)$$

$$\varphi_z = y^2e^z - 6xz^2 \xrightarrow[z]{\text{int wrt}} \varphi = y^2e^z - 2xz^3 + g(x, y)$$

$$\varphi = x^2\cos y - 2xz^3 + 3y + y^2e^z + c$$

any c will work. $c = 0$

$$\varphi = x^2 \cos y - 2xz^3 + 3y + y^2 e^z$$

$$\nabla \varphi = \vec{F}.$$