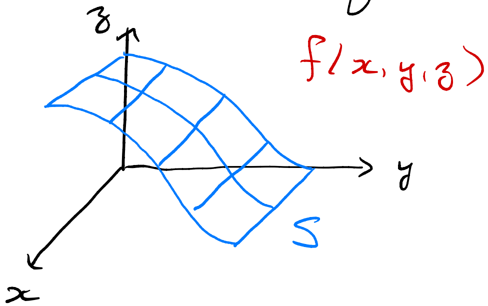
Lesson 20: Surface Integrals

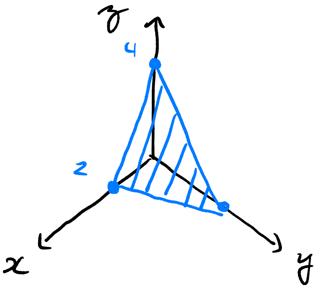


Surface integral:

$$\iint_S f(x,y,z)\,dS = \text{accumulation of } f(x,y,z) \text{ along } S.$$

if S is an object and f is the density of S, then $\iint_S f(x,y,z)\,dS$ is the mass of S.

eg. Parameterize the part of the plane $2x + y + z = 4$ in the first octant.



$$z = 4 - 2x - y$$

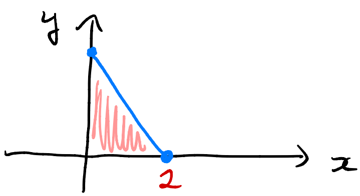
Choose parameters : $x = u$ $\quad y = v$

$$z = 4 - 2u - v$$

$$\vec{r}(u,v) = \langle u, v, 4 - 2u - v \rangle$$

What are bounds on u and v?

xy trace:

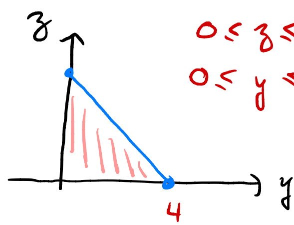


$$y = 4 - 2x$$

$$\begin{matrix} 0 \leq y \leq 4 - 2x \\ 0 \leq x \leq 2 \end{matrix} \rightsquigarrow \begin{matrix} 0 \leq v \leq 4 - 2u \\ 0 \leq u \leq 2 \end{matrix}$$

Another way: Solve for x,

yz trace: $z = 4 - y$



$0 \leq z \leq 4 - y$

$0 \leq y \leq 4$

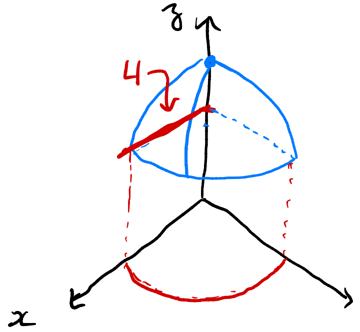
$$x = 2 - \frac{1}{2} y - \frac{1}{2} z$$

Choose param. $\quad y = u, \quad z = v$

So $\quad x = 2 - \frac{1}{2} u - \frac{1}{2} v$

$$\vec{r}(u,v) = \left\langle 2 - \frac{1}{2} u - \frac{1}{2} v, \ u, \ v \right\rangle \qquad \begin{matrix} 0 \leq v \leq 4 - u \\ 0 \leq u \leq 4 \end{matrix}$$

eg. Parametrize the surface $x^2+y^2+z^2=25$ in the first octant with $3 \leq z \leq 5$.

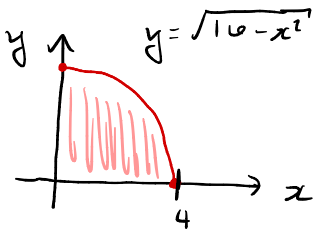


$$z = +\sqrt{25-x^2-y^2}$$

Choose parameters: $x = u \quad y = v$

So $\quad z = \sqrt{25-u^2-v^2}$

$$\vec{r}(u,v) = \left\langle u, v, \sqrt{25-u^2-v^2} \right\rangle$$



$$3 = z = \sqrt{25-x^2-y^2}$$

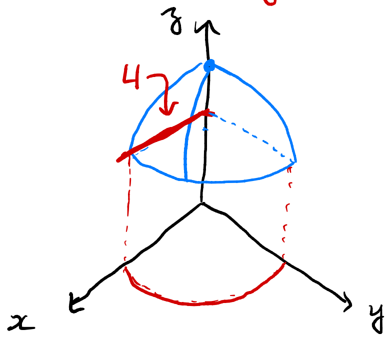
$$9 = 25 - x^2 - y^2$$

$$x^2+y^2 = 16$$

circle of radius 4

$$\begin{matrix} 0 \leq y \leq \sqrt{16-x^2} \\ 0 \leq x \leq 4 \end{matrix} \rightsquigarrow \begin{matrix} 0 \leq v \leq \sqrt{16-u^2} \\ 0 \leq u \leq 4 \end{matrix}$$

Another way:



Choose parameters:

$$x = u\cos(v) \qquad y = u\sin(v)$$

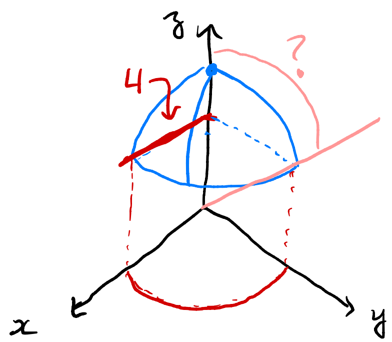
$$\begin{aligned} z &= \sqrt{25 - x^2 - y^2} \\ &= \sqrt{25 - u^2\cos^2 v - u^2\sin^2 v} \\ &= \sqrt{25 - u^2} \end{aligned}$$

$$\vec{r}(u,v) = \left\langle u\cos(v),\ u\sin(v),\ \sqrt{25 - u^2} \right\rangle$$

$$0 \leq v \leq \pi/2$$

$$0 \leq u \leq 4$$

Another way:



Choose parameters

$$x = 5\sin(v)\cos(u)$$

$$y = 5\sin(v)\sin(u)$$

$$\begin{aligned} z &= \sqrt{25 - x^2 - y^2} \\ &= \sqrt{25 - 25\sin^2(v)} \\ &= 5\sqrt{1 - \sin^2 v} \\ &= 5\sqrt{\cos^2 v} \\ &= 5\cos(v) \end{aligned}$$

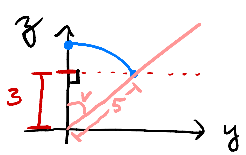
$\sin^2\theta + \cos^2\theta = 1$

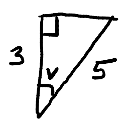
$\cos^2\theta = 1 - \sin^2\theta$

$$\vec{r}(u,v) = \left\langle 5\sin v\cos u,\ 5\sin v\sin u,\ 5\cos v \right\rangle$$

$$0 \leq u \leq \pi/2$$

$$0 \leq v \leq \cos^{-1}(3/5)$$

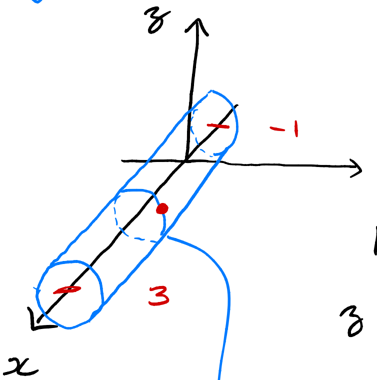




$$\cos(v) = \frac{3}{5}$$

$$v = \cos^{-1}(3/5)$$

eg. Parameterize $y^2 + z^2 = 4$ for $-1 \leq x \leq 3$.



Parameters:

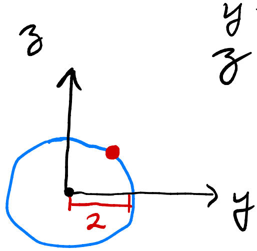
(1) location on x-axis u

(2) angle from y-axis v

Hence $x = u$

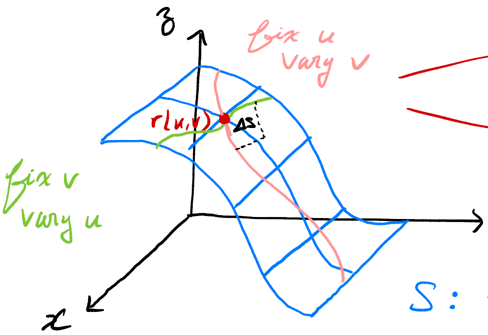
$y = 2\cos(v)$

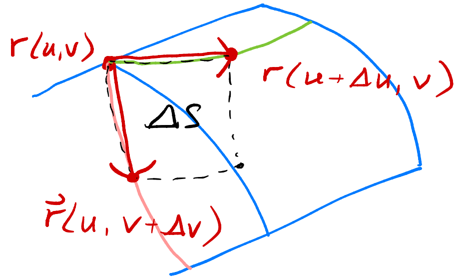
$z = 2\sin(v)$



$$\vec{r}(u,v) = \langle u, 2\cos v, 2\sin v \rangle$$

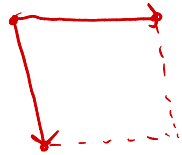
$$-1 \leq u \leq 3$$
$$0 \leq v \leq 2\pi$$





$S: \vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$
with domain of (u,v) D.

$\Delta S \approx$ area ($r(u,v+\Delta v) - r(u,v)$, $r(u+\Delta u,v) - r(u,v)$)



$$= \left| \big(r(u+\Delta u,v) - r(u,v)\big) \times \big(r(u,v+\Delta v) - r(u,v)\big) \right|$$

Recall:

$$\frac{\partial r}{\partial u} \approx \frac{r(u+\Delta u,v) - r(u,v)}{\Delta u}$$

$$\frac{\partial r}{\partial v} \approx \frac{r(u,v+\Delta v) - r(u,v)}{\Delta v}$$

$$r(u+\Delta u,v) - r(u,v) \approx \Delta u \frac{\partial r}{\partial u}$$

$$r(u,v+\Delta v) - r(u,v) \approx \Delta v \frac{\partial r}{\partial v}$$

$$\Delta S \approx \left| \Delta u \frac{\partial r}{\partial u} \times \Delta v \frac{\partial r}{\partial v} \right| = \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| \Delta u \Delta v$$

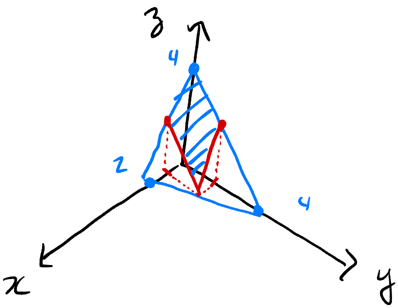
So as ΔS gets infinitely small

$$\Delta S \longrightarrow dS = \left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| du\, dv$$

Hence:

$$\iint_S f(x, y, z)\, dS = \iint_D f(x(u,v), y(u,v), z(u,v)) \left| r_u \times r_v \right| du\,dv$$

eg. Evaluate $\iint_S x+y\, dS$ where S is the plane $2x+y+z=4$ in the first octant above $0 \le x \le 1$ and $0 \le y \le 2$.



$$\vec{r}(u,v) = \langle u, v, 4-2u-v \rangle$$

$$D = \{ (u,v) \mid 0 \le u \le 1,\ 0 \le v \le 2 \}$$

$$r_u = \langle 1, 0, -2 \rangle$$

$$r_v = \langle 0, 1, -1 \rangle$$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0-(-2), -(-1-0), 1 \rangle$$

$$= \langle 2, 1, 1 \rangle$$

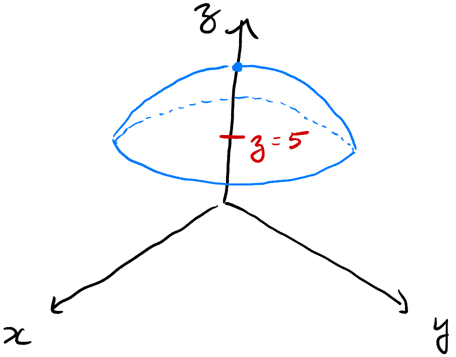
$$|r_u \times r_v| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$$

$$\iint_S x+y\, dS = \iint_D (u+v)\sqrt{6}\, du\, dv$$

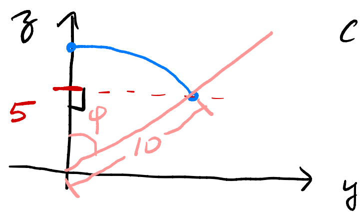
$$= \sqrt{6} \int_0^2 \int_0^1 u+v\, du\, dv$$

$$= 3\sqrt{6}$$

eg. Find the surface area of the cap of a sphere of radius 10 centered at the origin above $z = 5$.



$$\text{Surface Area} = \iint_S dS$$



$(0 \le \varphi \le \pi)$

$$\cos\varphi = \frac{5}{10}$$

$$\varphi = \cos^{-1}\left(\tfrac{1}{2}\right) = \pi/3$$

$$\vec{r}(\varphi, \theta) = \langle 10\sin\varphi\cos\theta,\ 10\sin\varphi\sin\theta,\ 10\cos\varphi \rangle$$

$$0 \le \theta \le 2\pi$$

$$0 \le \varphi \le \pi/3$$

$$r_\varphi \times r_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10\cos\varphi\cos\theta & 10\cos\varphi\sin\theta & -10\sin\varphi \\ -10\sin\varphi\sin\theta & 10\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \langle 0 + 100\sin^2\varphi\cos\theta,\ -(0 - 100\sin^2\varphi\sin\theta),\ 100\sin\varphi\cos\varphi\cos^2\theta + 100\sin\varphi\cos\varphi\sin^2\theta \rangle$$

$$= \langle 100\sin^2\varphi\cos\theta,\ 100\sin^2\varphi\sin\theta,\ 100\sin\varphi\cos\varphi \rangle$$

$$|r_\varphi \times r_\theta| = \sqrt{100^2\sin^4\varphi\cos^2\theta + 100^2\sin^4\varphi\sin^2\theta + 100^2\sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{\underbrace{100^2\sin^4\varphi + 100^2\sin^2\varphi\cos^2\varphi}}$$

$$= \sqrt{100^2\sin^2\varphi} = 100\sin\varphi$$

$$\iint_S dS = \int_0^{2\pi} \int_0^{\pi/3} 100 \sin\varphi \, d\varphi \, d\theta = 100\pi$$

Fact: For spheres centered at origin of fixed radius ρ, then

$$dS = \rho^2 \sin\varphi \, d\varphi \, d\theta$$