Lesson 21: Surface Integrals II

Last class: S surface parameterized by $\vec{r}(u,v)$ with domain D

$$\iint_S f(x,y,z)\,dS = \iint_R f(\vec{r}(u,v))\,|r_u \times r_v|\,dA$$

eg. Let S be the surface $z = f(x,y)$ over the region D in the xy-plane. Compute $\iint_S \rho(x,y)\,dS$.

$$S: \vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

$$r_x \times r_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$

$$|r_x \times r_y| = \sqrt{(f_x)^2 + (f_y)^2 + 1}$$

$$\iint_S \rho(x,y)\,dS = \iint_D \rho(x,y)\,|r_x \times r_y|\,dA$$

$$\boxed{\iint_S \rho(x,y)\,dS = \iint_D \rho(x,y)\sqrt{(f_x)^2 + (f_y)^2 + 1}\,dA}$$

e.g. Integrate $F(x,y) = x+y$ over the portion of the plane $x+y+z=6$ that lies above the square $-1 \le x \le 1$, $-1 \le y \le 1$ in the xy-plane.

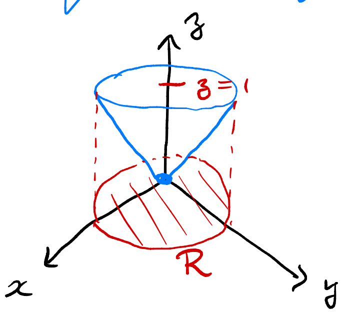
$$z = 6 - x - y$$

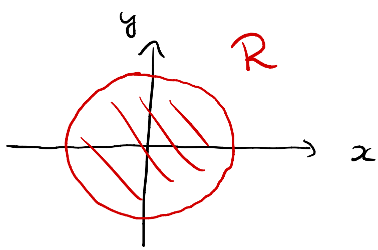
$$\iint_S x+y \, dS = \int_{-1}^{1} \int_{-1}^{1} (x+y)\sqrt{(-1)^2+(-1)^2+1} \, dy \, dx$$

$$= \sqrt{3} \int_{-1}^{1} \int_{-1}^{1} x+y \, dy \, dx$$

$$= 0$$

e.g. Find surface area of the cone $z^2 = x^2+y^2$ for $0 \le z \le 1$.





$$(1)^2 = x^2+y^2$$
$$1 = x^2+y^2$$

circle of radius 1

$$z = \sqrt{x^2+y^2} \qquad z_x = \frac{x}{\sqrt{x^2+y^2}} \qquad z_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$SA = \iint_S dS = \iint_R \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1} \, dA$$

$$= \iint_R \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\; dA$$

$$= \iint_R \sqrt{\frac{x^2+y^2}{x^2+y^2} + 1}\; dA$$

$$= \sqrt{2} \iint_R dA$$

$$= \pi\sqrt{2}$$

*On hw: $\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2+x^2} + a^2 \ln\left(x+\sqrt{a^2+x^2}\right)\right) + C$

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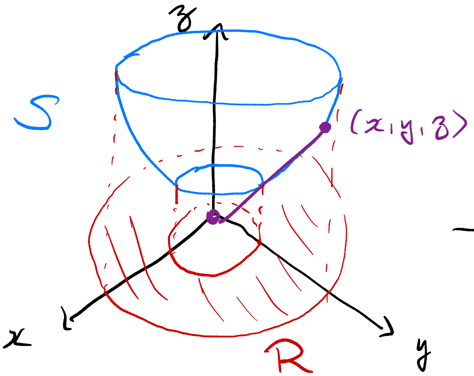
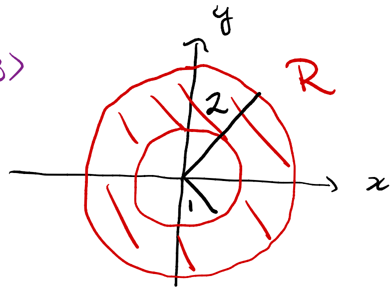
eg. Find the average distance from the origin to the surface $z = x^2 + y^2$, $1 \le z \le 4$.





$4 = x^2 + y^2$

$1 = x^2 + y^2$

$$AVG = \frac{\iint_S \text{distance to origin } dS}{\iint_S dS}$$

$$= \frac{\iint_S \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}\, dS}{\iint_S dS}$$

$$= \frac{\iint_S \sqrt{x^2 + y^2 + z^2}\, dS}{\iint_S dS}$$

$$\iint_S dS = \iint_R \sqrt{(2x)^2 + (2y)^2 + 1}\, dA$$

$$= \iint_R \sqrt{4(x^2 + y^2) + 1}\, dA$$

$$= \int_0^{2\pi} \int_1^2 \sqrt{4r^2 + 1}\; r\, dr\, d\theta$$

$$\approx 30.8$$

$$\iint_S \sqrt{x^2+y^2+z^2}\, dS$$

$$= \iint_R \sqrt{x^2+y^2+(x^2+y^2)^2}\,\sqrt{4(x^2+y^2)+1}\; dA$$

$$= \int_0^{2\pi}\int_1^2 \sqrt{r^2+r^4}\,\sqrt{4r^2+1}\; r\,dr\,d\theta$$

$$\approx 58.4$$

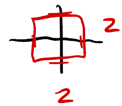
$$AVG = \frac{58.4}{30.8} \approx 1.9$$

eg. Find the avg. temp. of the surface $2x + 2y + z = 4$ over $|x| \le 1$ $|y| \le 1$ with temp. $T(x,y,z) = z^2$.

$\leadsto$ $-1 \le x \le 1$

$-1 \le y \le 1$

$$AVG = \frac{\iint_S T(x,y,z)\, dS}{\iint_S dS} = \frac{\iint_S z^2\, dS}{\iint_S dS}$$



$$z = 4 - 2x - 2y$$

$$\iint_S dS = \iint_R \sqrt{(-2)^2 + (-2)^2 + 1}\; dA$$

$$= 3 \iint_R dA = 3 \int_{-1}^{1} \int_{-1}^{1} dy\, dx$$

$$= 12$$

$$3 = \sqrt{(-2)^2 + (-2)^2 + 1}\; dA$$

$$\iint_S z^2\, dS = \iint_R (4 - 2x - 2y)^2 \cdot 3\, dA$$

$$= 3 \int_{-1}^{1} \int_{-1}^{1} (4 - 2x - 2y)^2\, dy\, dx$$

$$= 224$$

$$AVG = \frac{224}{12} = 18.\overline{6}$$