Lesson 24: Stoke's Thm pt II

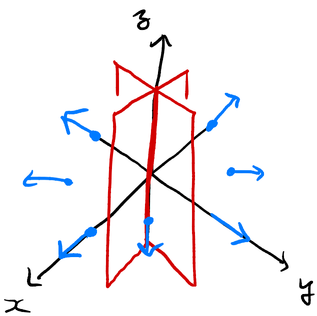
Recall: Given $\vec{F} = \langle f, g, h \rangle$ we define the curl of $\vec{F}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix}$$

Stoke's Thm: If S is a "good" surface with boundary C, then

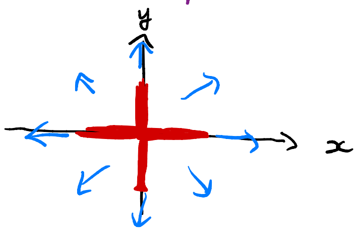
$$\iint_S \text{curl } \vec{F} \cdot \vec{n} \, dS = \int_C \vec{F} \cdot d\vec{r}$$

Consider $F = \langle x, y, 0 \rangle$



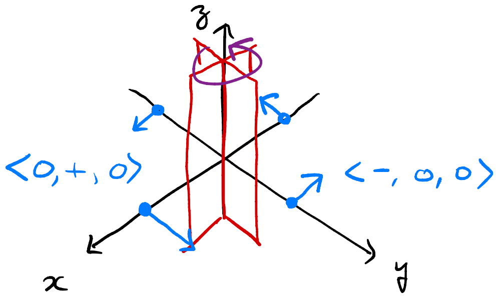
Imagine $\vec{F}$ is the current of some fluid and a paddle wheel at the origin aligned along the z-axis.

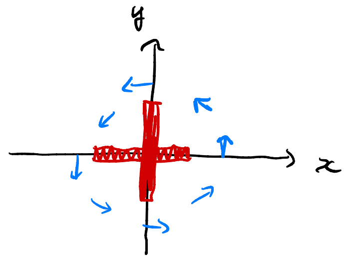
Would we expect the paddle to spin?



no.

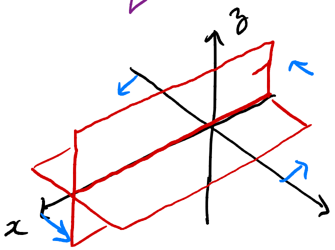
What about $F = \langle -y, x, 0 \rangle$





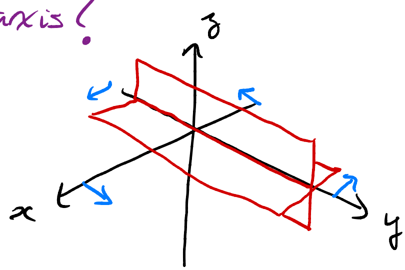
it will spin counter clockwise!

what if we align the paddle w/ the x-axis?



no spin.

y-axis?



no spin.

So < "spin" in x-axis, "spin" in y-axis, "spin" z-axis >

$= \langle 0, 0, + \rangle$

We call it the curl of F.

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \langle 0, 0, 1-(-1) \rangle$$

$$= \langle 0, 0, 2 \rangle$$

when $F = \langle x, y, 0 \rangle$

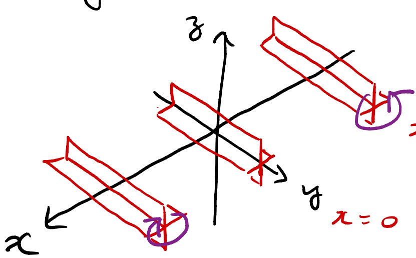
$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & 0 \end{vmatrix} = \langle 0, 0, 0 \rangle$$

What if $F = \langle 3, 0, x^2 \rangle$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3 & 0 & x^2 \end{vmatrix} = \langle 0, -(2x-0), 0 \rangle$$

$$= \langle 0, -2x, 0 \rangle$$

We only have spin if our axis is aligned in the y-axis with magnitude $2|x|$.



$x=-1$ $\quad \text{curl}\,F = \langle 0, 2, 0 \rangle$ Counter clockwise spin

$x=0$ $\quad \text{curl}\,F = \langle 0, 0, 0 \rangle$ no spin

$x=1$ $\quad \text{curl}\,F = \langle 0, -2, 0 \rangle$ clockwise spin.

Hence $\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS$

calculates the accumulation of spin of the paddle aligned with $\vec{n}$ all along S.

eg. Suppose $\vec{F}$ is conservative. Use Stokes thm to calculate the circulation of $\vec{F}$ along the boundary C of a surface S.

$\vec{F} = \nabla \varphi$

$$\int_C \vec{F} \cdot d\vec{r} \overset{S.T.}{=} \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS$$

$$= \iint_S \underbrace{\operatorname{curl}(\nabla \varphi)}_{0} \cdot \vec{n}\, dS$$

$$= 0$$

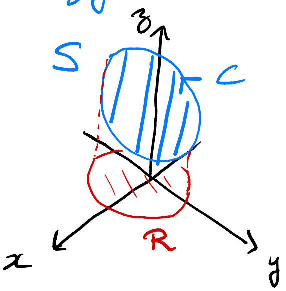
Also FTLI : $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla \varphi \cdot d\vec{r}$

Same pt!

$= \varphi(\text{end pt of } C) - \varphi(\text{initial pt of } C)$

$= 0$

eg. Fix some $0 \le \varphi \le \pi/2$.
Let S be a disk enclosed by the curve
$C: \vec{r}(t) = \langle \cos\varphi \cos t,\ \sin t,\ \sin\varphi \cos t \rangle$
for $0 \le t \le 2\pi$. Let $F = \langle -y, x, 0 \rangle$
Verify Stokes thm.



tilted disk

$r'(t) = \langle -\cos\varphi \sin t,\ \cos t,\ -\sin\varphi \sin t \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -\sin t, \cos\varphi \cos t, 0 \rangle \cdot \langle -\cos\varphi \sin t, \cos t, -\sin\varphi \sin t \rangle \, dt$$

$$= \int_0^{2\pi} \cos\varphi \sin^2 t + \cos\varphi \cos^2 t \; dt$$

$$= \cos\varphi \int_0^{2\pi} dt$$

$$= 2\pi \cos\varphi \quad \checkmark$$

Parametrize S: $\langle \cos\varphi \cos t,\ \sin t,\ \sin\varphi \cos t \rangle$ parametrizes boundary of S

$$s(r,t) = \langle r\cos\varphi \cos t,\ r\sin t,\ r\sin\varphi \cos t \rangle$$
$$0 \le t \le 2\pi$$
$$0 \le r \le 1$$

$$s_r \times s_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\varphi\cos t & \sin t & \sin\varphi\cos t \\ -r\cos\varphi\sin t & r\cos t & -r\sin\varphi\sin t \end{vmatrix}$$

$$= \langle -r\cos\varphi,\ 0,\ r\cos\varphi \rangle$$

$$\iint_S \operatorname{curl} F \cdot \vec{n}\, dS = \iint_R \langle 0, 0, 2\rangle \cdot \langle -r\cos\varphi, 0, r\cos\varphi\rangle\, dr\,dt$$

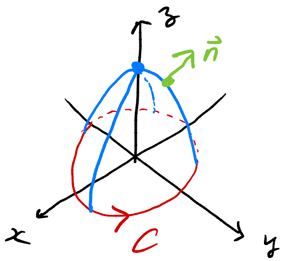
$$= \int_0^{2\pi}\int_0^1 2r\cos\varphi\, dr\,dt$$

$$= 2\pi\cos\varphi \int_0^1 2r\,dr$$

$$= 2\pi\cos\varphi$$

eg. Let S_a be the surface $z^2 = a^2(1-x^2-y^2)$ above the xy-plane with $a>0$ a constant. Let $\vec{n}$ point upwards and $F = \langle x-y, y-z, z-x \rangle$.

Find $\iint_{S_a} \text{curl}\,\vec{F}\cdot\vec{n}\, dS$ as a function of a.



xy trace:
$0 = a^2(1-x^2-y^2)$
$1 = x^2+y^2$

$$\iint_{S_a} \text{curl}\,\vec{F}\cdot\vec{n}\, dS \overset{S.T.}{=} \int_C \vec{F}\cdot d\vec{r} \overset{S.T.}{=} \iint_D \text{curl}\,\vec{F}\cdot\vec{n}\, dS$$

D is the unit disk in the xy-plane

Parameterize D: $s(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$
$0 \le r \le 1$
$0 \le \theta \le 2\pi$

$$\iint_{S_a} \text{curl}\,\vec{F}\cdot\vec{n}\, dS = \iint_D \text{curl}\,\vec{F}\cdot\vec{n}\, dS$$

$$= \int_0^{2\pi}\int_0^1 (\text{curl}\,\vec{F})(s)\cdot(s_r \times s_\theta)\, dr\, d\theta$$

$\text{curl}\, F = \langle 1,1,1\rangle \qquad s_r \times s_\theta = \langle 0,0,r\rangle$

$$= \int_0^{2\pi}\int_0^1 \langle 1,1,1\rangle \cdot \langle 0,0,r\rangle\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\, dr\, d\theta = 2\pi\int_0^1 r\, dr = \pi.$$