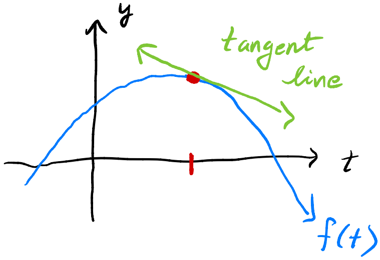
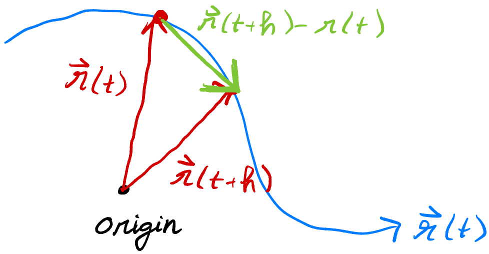
Lesson 4: Calculus of vector valued functions.

Recall: Derivate of scalar functions



$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$



Derivative of a vector function.

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$r(t) = \langle x(t),\ y(t),\ z(t) \rangle$$

$$r'(t) = \lim_{h\to 0} \frac{1}{h}\Big(\langle x(t+h), y(t+h), z(t+h) \rangle - \langle x(t), y(t), z(t) \rangle \Big)$$

$$= \lim_{h\to 0} \left\langle \frac{x(t+h)-x(t)}{h},\ \frac{y(t+h)-y(t)}{h},\ \frac{z(t+h)-z(t)}{h} \right\rangle$$

$$= \langle x'(t),\ y'(t),\ z'(t) \rangle \quad \checkmark$$

When $\overbrace{\vec{r}\,'(t) \neq \vec{0}}^{\vec{r} \text{ is smooth at } t.}$ we say that $\vec{r}\,'(t)$ is the tangent vector to $r(t)$ at t.

In this case $r'(t)$ may happen to not be a unit vector.

Unit tangent vector: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

e.g. Find unit tangent vector $\vec{T}(t)$ of $\vec{r}(t) = \langle \cos t, \sin t, t^2 \rangle$ at time $t = \pi/2$.

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 2t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (2t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + 4t^2}$$

$$= \sqrt{4t^2 + 1}$$

$$\vec{T}(t) = \frac{1}{\sqrt{4t^2+1}} \langle -\sin t, \cos t, 2t \rangle$$

$$= \left\langle \frac{-\sin t}{\sqrt{4t^2+1}}, \frac{\cos t}{\sqrt{4t^2+1}}, \frac{2t}{\sqrt{4t^2+1}} \right\rangle$$

$$\vec{T}\left(\frac{\pi}{2}\right) = \left\langle \frac{-1}{\sqrt{\pi^2+1}}, 0, \frac{\pi}{\sqrt{\pi^2+1}} \right\rangle$$

Integration of $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

e.g. Evaluate $\int_0^2 e^t \langle 1, t, t^2 \rangle \, dt$

$$= \int_0^2 \langle e^t, te^t, t^2e^t \rangle \, dt$$

$$= \left\langle \int_0^2 e^t dt, \int_0^2 te^t dt, \int_0^2 t^2e^t \, dt \right\rangle$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$= \langle e^2 - 1, e^2 + 1, 2e^2 - 2 \rangle$$

$$\int_0^2 te^t dt = te^t\Big|_0^2 - \int_0^2 e^t dt = 2e^2 - (e^2 - 1)$$

$u = t$ $\quad du = dt$

$dv = e^t dt$ $\quad v = \int e^t dt = e^t$

$$\int_0^2 t^2 e^t dt = t^2e^t\Big|_0^2 - 2\int_0^2 te^t \, dt$$

$u = t^2$ $\quad du = 2t\,dt$

$dv = e^t dt$ $\quad v = e^t$

$$= 4e^2 - 0 - 2(e^2 + 1) = 2e^2 - 2$$

Motion in Space part I

Consider $\vec{r}(t)$ as describing the position of some object in space. We call it the position vector.

$\vec{v}(t) = r'(t)$ velocity vector

$\vec{a}(t) = \vec{v}'(t) = r''(t)$ acceleration vector

The magnitude (absolute value) of $v(t)$ is the speed.

$$\text{speed} = |v(t)|$$

e.g. If the acceleration of an object is $\vec{a}(t) = \langle 1, t, t^2 \rangle$ and the initial velocity is $\langle 1, 0, 0 \rangle$ and the initial position is $\langle 1, 2, 3 \rangle$, then $\vec{r}(t)$.

First find
$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t)\,dt \\ &= \int \langle 1, t, t^2 \rangle dt \\ &= \left\langle \int 1\,dt, \int t\,dt, \int t^2\,dt \right\rangle \\ &= \langle t + c_1, t^2/2 + c_2, t^3/3 + c_3 \rangle\end{aligned}$$

$$\langle 1, 0, 0 \rangle = \vec{v}(0) = \langle 0 + c_1, 0 + c_2, 0 + c_3 \rangle$$

$$\langle 1, 0, 0 \rangle = \langle c_1, c_2, c_3 \rangle$$

$$c_1 = 1 \qquad c_2 = 0 \qquad c_3 = 0$$

$$\vec{v}(t) = \left\langle t+1, \frac{t^2}{2}, \frac{t^3}{3} \right\rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \int t+1\,dt,\ \int \frac{t^2}{2}\,dt,\ \int \frac{t^3}{3}\,dt \right\rangle$$

$$= \left\langle \frac{t^2}{2}+t,\ \frac{t^3}{6},\ \frac{t^4}{12} \right\rangle + \vec{C}$$

$$\vec{C} = \langle c_1, c_2, c_3 \rangle$$

$$= \left\langle \frac{t^2}{2}+t+c_1,\ \frac{t^3}{6}+c_2,\ \frac{t^4}{12}+c_3 \right\rangle$$

$$\langle 1, 2, 3 \rangle = \vec{r}(0) = \left\langle \frac{0^2}{2}+0,\ \frac{0^3}{6},\ \frac{0^4}{12} \right\rangle + \vec{C}$$

$$\langle 1, 2, 3 \rangle = \vec{C}$$

$$\vec{r}(t) = \left\langle \frac{t^2}{2}+t,\ \frac{t^3}{6},\ \frac{t^4}{12} \right\rangle + \langle 1, 2, 3 \rangle$$

$$= \left\langle \frac{t^2}{2}+t+1,\ \frac{t^3}{6}+2,\ \frac{t^4}{12}+3 \right\rangle$$

e.g. Given $\vec{a}(t) = \langle 7t, e^{-t}, 5 \rangle$ and $\vec{v}(0) = \langle 1, 1, 1 \rangle$ and $\vec{r}(0) = \langle 2, 0, 0 \rangle$. Find $\vec{r}(t)$.

Step 1: $\vec{v}(t) = \int \vec{a}(t)\, dt$

$$= \int \langle 7t, e^{-t}, 5 \rangle\, dt$$

$$= \langle \int 7t\,dt, \int e^{-t}dt, \int 5\,dt \rangle$$

$$= \langle \tfrac{7}{2}t^2, -e^{-t}, 5t \rangle + \vec{c}$$

$$\langle 1, 1, 1 \rangle = v(0) = \langle 0, -1, 0 \rangle + \vec{c}$$

$$\vec{c} = \langle 1, 1, 1 \rangle - \langle 0, -1, 0 \rangle$$

$$= \langle 1, 2, 1 \rangle$$

$$\vec{v}(t) = \langle \tfrac{7}{2}t^2 + 1, -e^{-t} + 2, 5t + 1 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\, dt$$

$$= \langle \int \tfrac{7}{2}t^2 + 1\, dt, \int -e^{-t} + 2\, dt, \int 5t + 1\, dt \rangle$$

$$= \langle \tfrac{7}{6}t^3 + t, e^{-t} + 2t, \tfrac{5}{2}t^2 + t \rangle + \vec{c}$$

$$\langle 2, 0, 0 \rangle = \vec{r}(0) = \langle 0, 1, 0 \rangle + \vec{c}$$

$$\vec{c} = \langle 2, 0, 0 \rangle - \langle 0, 1, 0 \rangle$$

$$= \langle 2, -1, 0 \rangle$$

$$\vec{r}(t) = \langle \tfrac{7}{6}t^3 + t + 2, e^{-t} + 2t - 1, \tfrac{5}{2}t^2 + t \rangle$$