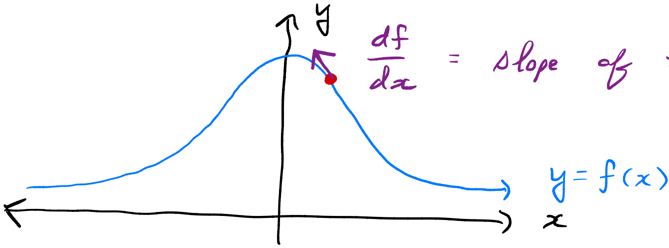
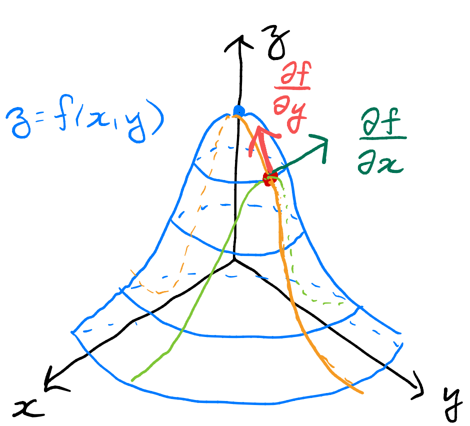
Lesson 7 : Partial Derivatives



$\frac{df}{dx}$ = slope of the tangent line.



Partial Derivatives

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

eg. Let $f(x, y) = x^2 + y^3 + 2xy^2$, find f_x f_y.

For f_x treat y as a constant and diff. wrt x

$$f_x = \frac{\partial f}{\partial x} = 2x + 0 + 2y^2$$

For f_y treat x as a constant and diff wrt y

$$f_y = \frac{\partial f}{\partial y} = 0 + 3y^2 + 4xy$$

eg. Let $z = x \ln(e^x + y^2)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial x} = (1)\ln(e^x + y^2) + x\,\frac{1}{e^x + y^2}\,(e^x + 0)$$

$$\frac{\partial z}{\partial y} = x \cdot \frac{1}{e^x + y^2} \cdot (0 + 2y)$$

Given $f(x)$ we higher derivatives

$$f \xrightarrow{\frac{d}{dx}} f' \xrightarrow{\frac{d}{dx}} f'' \xrightarrow{\frac{d}{dx}} f''' \longrightarrow \cdots$$

Now given $f(x,y)$ we higher partial derivatives

$$\begin{tikzcd}[row sep=small]
 & & f_{xx} \arrow[r] \arrow[dr] & f_{xxx} \; \cdot \\
 & f_x \arrow[ur, "\partial/\partial x"] \arrow[dr, "\partial/\partial y"'] & & f_{xxy} \; \cdot \\
 & & f_{xy} \arrow[r] \arrow[dr] & f_{xyx} \; \cdot \\
f \arrow[uur, "\partial/\partial x"] \arrow[ddddr, "\partial/\partial y"'] & & & f_{xyy} \; \cdot \\
 & & f_{yx} \arrow[r] \arrow[dr] & f_{yxx} \; \cdot \\
 & & & f_{yxy} \; \cdot \\
 & f_y \arrow[uur, "\partial/\partial x"] \arrow[dr, "\partial/\partial y"'] & & f_{yyx} \; \cdot \\
 & & f_{yy} \arrow[ur] \arrow[r] & f_{yyy} \; \cdot
\end{tikzcd}$$

partials like f_{xy} and f_{yx} are called mixed partial derivatives

e.g. Given $f(x,y) = 5x^3y^2 + 4y^5 + 3x - y$,
find f_{xx} f_{xy} f_{yx} f_{yy}

$f_x = 15x^2y^2 + 0 + 3 + 0 \longrightarrow f_{xx} = 30xy^2$

$\longrightarrow f_{xy} = 30x^2y$

$f_y = 10x^3y + 20y^4 + 0 - 1 \longrightarrow f_{yx} = 30x^2y$

$\longrightarrow f_{yy} = 10x^3 + 80y^3$

Fact: In good situations (like this class) $f_{xy} = f_{yx}$.

$(f_{xyx} = f_{xxy} = f_{yxx})$

order of the partial doesnt matter.

e.g. If $f(x,y,z) = xyz$

$\frac{\partial f}{\partial x} = yz$ $\qquad \frac{\partial^2 f}{\partial y \partial x} = z$ $\qquad \frac{\partial^3 f}{\partial z \partial y \partial x} = 1$

$\frac{\partial f}{\partial y} = xz$ $\qquad \frac{\partial^2 f}{\partial z \partial y} = x$ $\qquad \frac{\partial^3 f}{\partial x \partial z \partial y} = 1$

$\frac{\partial f}{\partial z} = xy$ $\qquad \frac{\partial^2 f}{\partial x \partial z} = y$ $\qquad \frac{\partial^3 f}{\partial y \partial x \partial z} = 1$

Chain rule

Recall : $y = f(x)$ and $x = g(t)$

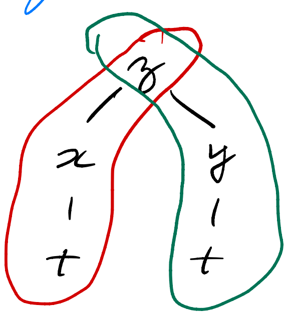
then $y = f(g(t))$ and $\boxed{\dfrac{dy}{dt} = \dfrac{df}{dx} \cdot \dfrac{dx}{dt}}$

Now : $z = f(x,y)$ $x = g(t)$ and $y = h(t)$

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$

eg Consider $z = x + y^2$ $x = e^t$ $y = \ln(t)$

find dz/dt



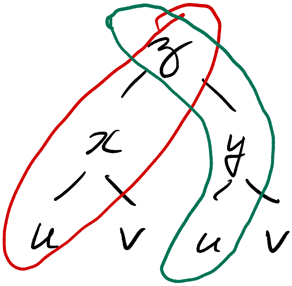
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (1)(e^t) + (2y)\left(\frac{1}{t}\right)$$

$$= e^t + \frac{2\ln(t)}{t}$$

$$z(t) = e^t + (\ln(t))^2$$

e.g. Consider $z = \sin(x+y)$ $\quad x = u^2 + v \quad y = 1 - 2uv$.
find $\partial z / \partial u$.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

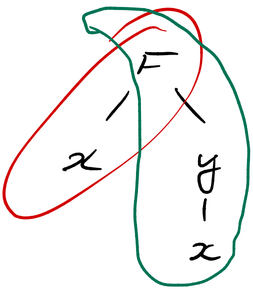
$$= (\cos(x+y))(2u) + (\cos(x+y))(-2v)$$

$$= (2\cos(x+y))(u-v)$$

$$= 2(u-v)\cos(u^2+v+1-2uv) \quad \checkmark$$

eg Consider $x^2 + 2y^2 = 4$, suppose y is a function of x, find dy/dx.

$$F = x^2 + 2y^2 - 4 = 0$$



$$0 = \frac{dF}{dx} = F_x + F_y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{2x + 0 + 0}{0 + 4y + 0}$$

$$= \frac{-2x}{4y}$$

$$= -\frac{x}{2y}$$

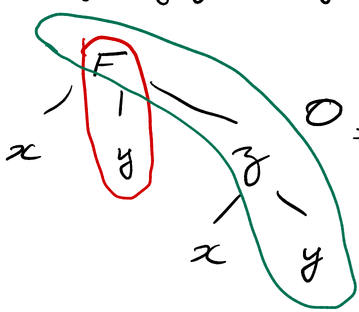
eg. (a) $3x^2 + 2x^2y = 4y^3$, y function of x
find dy/dx.

(b) $xy + yz + xz = 3$, z is a function of x and y, find $\partial z/\partial y$.

a) $F = 3x^2 + 2x^2y - 4y^3 = 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{6x + 4xy + 0}{0 + 2x^2 - 12y^2}$$

b) $F = xy + yz + xz - 3 = 0$



$$0 = \frac{\partial F}{\partial y} = F_y + F_z \cdot \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x + z + 0 + 0}{0 + y + x + 0} = -\frac{x+z}{y+x}$$

e.g. If $f(x,y) = \ln(3x^2 + 7y^2 + 7)$, find f_{xx} f_{xy}

$$f_x = \frac{6x}{3x^2 + 7y^2 + 7}$$

$\partial/\partial x$

Quotient rule

$$f_{xx} = \frac{6(3x^2+7y^2+7) - 6x(6x)}{(3x^2+7y^2+7)^2}$$

$\partial/\partial y$

$$f_{xy} = \frac{-6x(14y)}{(3x^2+7y^2+7)^2}$$