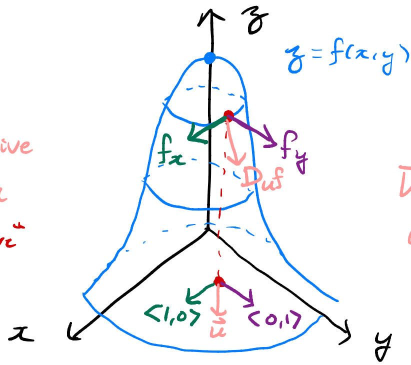
Lesson 8: Directional Derivative and Gradient

Last class:

$D_{\vec{u}} f$ is the derivative in the $\vec{u}$ direction
"directional derivative"



$\vec{u} = \langle u_1, u_2 \rangle$
unit vector

$$D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x + h u_1, y + h u_2) - f(x,y)}{h}$$

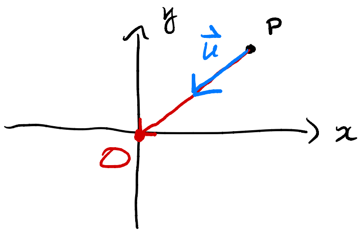
$\vec{u} = \langle 1, 0 \rangle \qquad D_{\vec{u}} f = f_x$

$\vec{u} = \langle 0, 1 \rangle \qquad D_{\vec{u}} f = f_y$

In fact for unit vector $\vec{u}$

$$\begin{aligned} D_{\vec{u}} f &= u_1 D_{\langle 1,0 \rangle} f + u_2 D_{\langle 0,1 \rangle} f \\ &= u_1 f_x + u_2 f_y \\ &= \langle f_x, f_y \rangle \cdot \vec{u} \end{aligned}$$

eg. Let $f(x,y) = e^{2x+y}$. Find the directional derivative towards the origin at $P(1,1)$.



$$\vec{PO} = \langle 0-1,\ 0-1 \rangle = \langle -1, -1 \rangle$$

$$\vec{u} = \frac{\vec{PO}}{|\vec{PO}|} = \left\langle \frac{-1}{\sqrt{2}},\ \frac{-1}{\sqrt{2}} \right\rangle$$

$$\begin{aligned}
D_{\vec{u}} f &= \langle f_x, f_y \rangle \cdot \vec{u} \\
&= \langle 2e^{2x+y},\ e^{2x+y} \rangle \cdot \langle -1/\sqrt{2},\ -1/\sqrt{2} \rangle \\
&= \frac{-2}{\sqrt{2}} e^{2x+y} - \frac{1}{\sqrt{2}} e^{2x+y} \\
&= \frac{-3}{\sqrt{2}} e^{2x+y}
\end{aligned}$$

$D_u f(1,1) = -\dfrac{3}{\sqrt{2}} e^3$ is the rate of change of f in the direction $\vec{u}$.

$\langle f_x, f_y \rangle$ is the gradient of $f(x,y)$

$$\text{grad}(f) = \vec{\nabla} f = \nabla f = \langle f_x, f_y \rangle$$

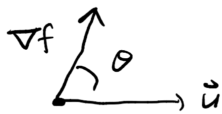
Aside: if $f(x,y,z)$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

What does ∇f mean geometrically?

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$
$$= |\nabla f| |\vec{u}| \cos\theta$$
$$= |\nabla f| \underbrace{\cos\theta}_{-1 \le \quad \le 1}$$



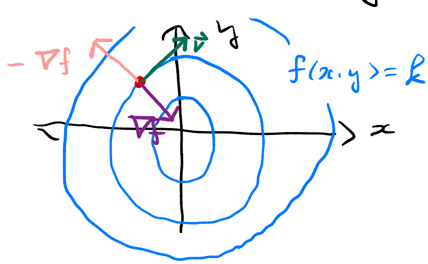
$D_{\vec{u}} f$ is maximized when $\theta = 0$

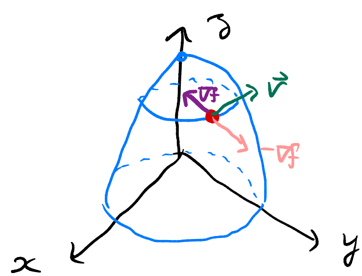
Thus ∇f points in the direction of
steepest ascent

Rate of change of f in the $\vec{u}$ direction
is 0 when $\vec{u}$ and ∇f are perpendicular.

eg Consider $f(x,y) = 3 - 2x^2 - y^2$. From $P(-1,1)$
find the direction of steepest ascent
steepest descent, no change.

$$\nabla f = \langle -4x, -2y \rangle$$





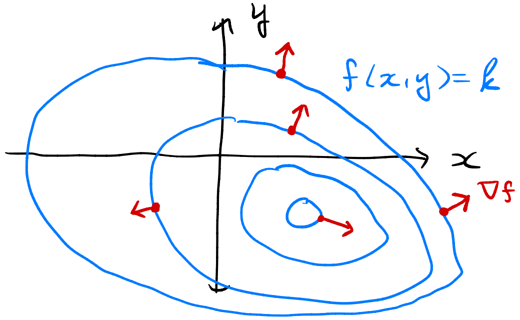
$\nabla f(-1,1) = \langle 4, -2 \rangle$ steepest ascent
$-\nabla f(-1,1) = \langle -4, 2 \rangle$ steepest descent.

$$\vec{v} \cdot \nabla f(-1,1) = 0$$
$$\langle v_1, v_2 \rangle \cdot \langle 4, -2 \rangle = 0$$
$$4v_1 - 2v_2 = 0$$

$$v_2 = 2v_1 = 2t$$
$$v_1 = t \qquad t \neq 0$$

$t = 1$ $\quad \vec{v} = \langle 1, 2 \rangle$ direction of no change.

In general, given $f(x,y)$ is ∇f normal to the level curves?



parameterize a level curve $f(x,y) = k$

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

$$f(x(t), y(t)) = k$$

$$F = f(x,y) - k$$

$$\begin{array}{ccc} & F & \\ / & & \backslash \\ x & & y \\ | & & | \\ t & & t \end{array} \qquad 0 = \frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

$$0 = f_x \cdot x'(t) + f_y\, y'(t)$$
$$0 = \nabla f \cdot \underline{r'(t)}$$

tangent to r, so tangent to level curve

Yes, ∇f is normal to $f(x,y) = k$.

Also (suppose y is a fn of x)

$$\begin{tikzcd}[column sep=small] & F \arrow[dl, dash] \arrow[dr, dash] & \\ x & & y \arrow[d, dash] \\ & & x \end{tikzcd}$$

$$0 = \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx}$$

$$0 = f_x + f_y \, {dy}/{dx}$$

$$dy/dx = -f_x/f_y$$

slope of the level curve.

So we can now construct vectors $\vec{v}$ that are tangent to level curve.

$\vec{v}$ must have slope $dy/dx = -f_x/f_y$

$\vec{v} = t\langle 1, -f_x/f_y \rangle \qquad t \neq 0$

take $t = 1 \qquad \vec{v} = \langle 1, -f_x/f_y \rangle$

So previous ex. $\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{-(-4(-1))}{-2(1)} = \frac{-4}{-2} = 2$

$\vec{v} = \langle 1, 2 \rangle$

eg. Let $f(x,y,z) = xyz$. Find equation of a plane tangent to the level surface containing $(1,2,3)$.

$f(1,2,3) = 1 \cdot 2 \cdot 3 = 6$

level surface $f(x,y,z) = 6$

need a point : $(1,2,3)$

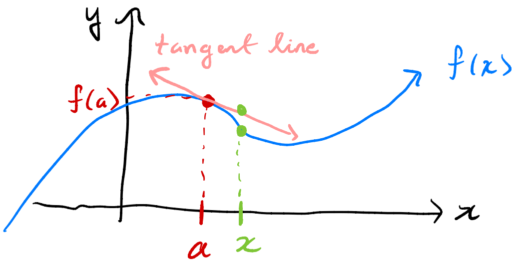
need a vector normal to the plane : $\vec{n}$

$$\begin{aligned}\vec{n} &= \nabla f(1,2,3) \\ &= \langle f_x, f_y, f_z \rangle \big|_{\langle 1,2,3 \rangle} \\ &= \langle yz, xz, xy \rangle \big|_{\langle 1,2,3 \rangle} \\ &= \langle 6, 3, 2 \rangle\end{aligned}$$

plane : $\vec{n} \cdot \langle x-1, y-2, z-3 \rangle = 0$

$$\boxed{6x + 3y + 2z = 18}$$

Tangent Planes and Linear Approximations



Given surface $z = f(x, y)$ to linearly approximate f at (x_0, y_0, z_0) we will use a tangent plane.

Idea: view the surface $f(x,y)$ as single level surface of some 4D object called $F(x,y,z)$. Then $\vec{\nabla} F$ is normal to the level surfaces.

$$F(x,y,z) = f(x,y) - z$$

level surface : $F = 0$ $\quad f(x,y) - z = 0$

$$z = f(x,y)$$

∇F is normal to $F = 0$ which is $z = f(x,y)$

$$\nabla F = \langle F_x, F_y, F_z \rangle$$
$$= \langle f_x, f_y, -1 \rangle$$

$$\boxed{\vec{n} = \nabla F(x_0, y_0, z_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle}$$

Tangent Plane: $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

eg. Find equation for tangent plane to

$f(x,y) = e^{x^2 y}$ at $(2, 1, e^4)$.

$$f_x = 2xy e^{x^2 y} \qquad f_y = x^2 e^{x^2 y}$$

$$\vec{n} = \langle f_x(2,1), f_y(2,1), -1 \rangle$$
$$= \langle 4e^4, 4e^4, -1 \rangle$$

$$\vec{n} \cdot \langle x-2, y-1, z-e^4 \rangle = 0$$

$$\boxed{z = e^4 + 4e^4(x-2) + 4e^4(y-1)}$$

eg. From previous linearly approx $f(2.1, 0.95)$.

$$f(2.1, 0.95) \approx e^4 + 4e^4(2.1-2) + 4e^4(0.95-1)$$
$$\approx 65.518$$

$$f(2.1, 0.95) = 65.990 \ldots$$

$$\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\underbrace{(z - z_0)}_{\Delta z} = f_x \underbrace{(x - x_0)}_{\Delta x} + f_y \underbrace{(y - y_0)}_{\Delta y}$$

$$dz = f_x \, dx + f_y \, dy$$

eg. Let $z = f(x,y) = x^2 y$. Approximate % change of z if x is increased by 1%
y is decreased by 3%.

$$dz = f_x\, dx + f_y\, dy$$
$$dz = 2xy\, dx + x^2 dy$$

dx, dy, dz absolute changes

$\frac{dx}{x}$, $\frac{dy}{y}$, $\frac{dz}{z}$ are relative or percent changes.

$$\frac{dz}{z} = \frac{2xy}{z} dx + \frac{x^2}{z} dy$$
$$= \frac{2xy}{x^2 y} dx + \frac{x^2}{x^2 y} dy$$
$$= 2 \frac{dx}{x} + \frac{dy}{y}$$
$$\frac{dz}{z} = 2(0.01) + (-0.03)$$
$$= -0.01$$

z decreases by 1% .