Varieties

In order to awnser questions about solutions to polynomial equations we will need an environment to do geometry. For the uniniated, geometry typically refers to working in a metric space. Common examples that we will take inspiration from are $\mathbb{R}^n$ and $\mathbb{C}^n$.

Def. Let k be a field and $n \in \mathbb{N}$ we define the affine n-space over k as

$$\mathbb{A}^n_k := \mathbb{A}^n = \{ x = (x_1, \dots, x_n) \in k^n \}$$

The reader may notice that as sets $\mathbb{A}^n_k$ is the same as k^n. The reason for this distinction is that k^n carries with it a k-vector space structure while we will latter see that the topology we endow $\mathbb{A}^n_k$ with is quite strange at first.

One of our main objectives for developing this language is convert problems in geometry to those in algebra.

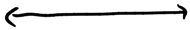
Def. Let k be a field and $T_1, \dots, T_n$ trancendental over k, then we set $A := k[T_1, \dots, T_n]$

Given $S \subset A$ we define an algebraic set to be $V \subset \mathbb{A}^n_k$ such that $V = V(S) := \{ x \in \mathbb{A}^n_k \mid P(x) = 0 \ \ \forall P \in S \}$

and the ideal corresponding to an alg. set $V \subset \mathbb{A}^n_k$ to be $I(V) := \{ P(T) \in A \mid P(x) = 0 \ \ \forall x \in V \}$

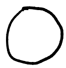
Ex 1) $A = \mathbb{C}[T_1, T_2]$

a) line : $V = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} \mid x_2 = 0\} = V(T_2)$



$$I(V) = T_2 A$$

b) circle : $V = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} \mid x_1^2 + x_2^2 = 1\} = V(T_1^2 + T_2^2 - 1)$



$$I(V) = T_1^2 + T_2^2 - 1)A$$

c) parabola : $V = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} \mid x_1 - x_2^2 = 0\} = V(T_1 - T_2^2)$

$$I(V) = (T_1 - T_2^2)A$$

d) intersecting lines : $V = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} \mid x_1^2 + x_2^2 = 0\} = V(T_1^2 + T_2^2)$

$$I(V) = (T_1^2 + T_2^2)A$$

e) parallel line : $V = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} \mid x_2^2 = 1\} = V(T_2^2 - 1)$

$$I(V) = (T_2^2 - 1)A$$

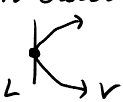
f) double line : $V = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} \mid x_2^2 = 0\} = V(T_2^2)$

$$I(V) = T_2 A$$ note this is same ideal as the line :)

g) point : $V = \{(0,0)\} = V(T_1, T_2)$

$$I(V) = (T_1, T_2)A$$

h) Intersection of a parabola and a line



$L = V(x_1)$ $\quad W = V(T_1 - T_2^2)$

$$L \cap V = \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} \mid x_1 = 0,\ x_1 - x_2^2 = 0\}$$
$$= \{(x_1, x_2) \in \mathbb{A}^2_{\mathbb{C}} \mid x_1 = 0,\ x_2^2 = 0\}$$
$$= V(T_1, T_2^2)$$

$$I(L \cap V) = (T_1, T_2)A$$ again same ideal as the point :)

i) Given $\mathfrak{a} \subset A$ ideal and $S \subset A : \langle S \rangle = \mathfrak{a}$, then $V(\mathfrak{a}) = V(S)$. In particular if $S = \{f_1, \dots, f_m\} \Rightarrow V(\mathfrak{a}) = V(f_1, \dots, f_m)$

Def An **aff. variety** is an algebraic set V such that $I(V)$ prime.

Ex 2) In example 1 only d and e are not varieties since $T_1^2 + T_2^2 = (T_1 - iT_2)(T_1 + iT_2)$
$T_2^2 - 1 = (T_2 - 1)(T_2 + 1)$

$$\begin{array}{ccc} \text{Geometry} & & \text{Algebra} \\ \left\{ \begin{array}{c} \text{algebraic} \\ \text{sets } V \subset \mathbb{A}^n \end{array} \right\} & \begin{array}{c} \xrightarrow{I(\cdot)} \\ \xleftarrow[V(\cdot)]{} \end{array} & \left\{ \begin{array}{c} \text{radical ideals} \\ \text{of } A \end{array} \right\} \\ \left\{ \begin{array}{c} \text{affine} \\ \text{variety } V \subset \mathbb{A}^n \end{array} \right\} & \begin{array}{c} \xrightarrow{I(\cdot)} \\ \xleftarrow[V(\cdot)]{} \end{array} & \left\{ \begin{array}{c} \text{prime ideals} \\ \text{of } A \end{array} \right\} \end{array}$$

Easy fact: Given an alg. set W, $V(I(W)) = W$
Given $\mathfrak{a} \subset A$ ideal, $I(V(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}$

This is the framework which alows us to convert geometry to algebra and algebra to geometry.

In order to delve deeper into this relationship we will need the following ubiquitous theorems of Hilbert.

Thm (Basis Thm) Ideals of A are finitely generated.

Thm (Nullstellensatz) Let k be an algebraically closed field and $\mathfrak{a} \subset A$ an ideal then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

The basis thm $\Rightarrow$ aff. varieties are zero sets to finitely many polys in A

The Nullstellensatz gives us the following inclusion reversing bijections (if k alg. closed)

$$\left\{\begin{matrix}\text{alg sets}\\ V \subset \mathbb{A}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{radical ideals}\\ \text{of } A\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{aff var.}\\ V \subset \mathbb{A}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{prime ideals}\\ \text{of } A\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{points of}\\ x \in \mathbb{A}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{maximal ideals}\\ \text{of } A\end{matrix}\right\}$$

$$(x_1, \ldots, x_n) \in A \longmapsto (T_1 - x_1, \ldots, T_n - x_n)$$

Zariski Topology

One of the reasons we distinguish $\mathbb{A}^n_k$ and k^n is the following topology we endow $\mathbb{A}^n_k$ with.

Prop 1) $\emptyset = V(1)$, $\mathbb{A}^n = V(0)$

2) $\mathfrak{a}, \mathfrak{b} \subset A$ ideals $\Rightarrow V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$

3) $\{\mathfrak{a}_i\}_{i\in I}$ a family of ideals of A

$$\bigcap_{i\in I} V(\mathfrak{a}_i) = V\left(\textstyle\sum_{i\in I} \mathfrak{a}_i\right)$$

This prop implies that the alg. sets of $\mathbb{A}^n$ satisfy the axioms of closed sets. We call the resulting topology the Zariski Topology

Def. Let X be a topological space, X is called reducible if there exists two proper closed subsets Z_1 and Z_2 such that $X = Z_1 \cup Z_2$. If X is not reducible we say it is irreducible.

Prop An algebraic set is an affine variety iff it is irreducible.

Prop Open sets in an irreducible topological space are dense.

Since $\mathbb{A}^n = V(0)$ and $I(V(0)) = 0$ is prime in A, $\mathbb{A}^n$ is an affine variety. The first prop implies $\mathbb{A}^n$ is irred so the second prop implies all open subsets of $\mathbb{A}^n$ are dense. If $k = \mathbb{C}$, then we see that the topology on $\mathbb{A}^n_{\mathbb{C}}$ is strictly coarser than the Euclidean top. on $\mathbb{C}^n$. Open sets in $\mathbb{C}^n$ can be made arbitrarily small, but in $\mathbb{A}^n_{\mathbb{C}}$ open sets are dense. The fact that open sets in $\mathbb{A}^n_k$ are so big alows us to develope a diverse local theory.

Subvarieties

Up until this point our varieties have been subsets of affine n-space with corresponding ideals lying in A. Using the Zariski topology on $\mathbb{A}^n$ we can convert our algebra – geometry map to a relative version.

Def. Let $V \subset \mathbb{A}^n$ be an algebraic set define the ring of polynomials on V to be
$k[V] = A / I(V)$

The reason for the language "ring of polynomials on V" comes from the following:

let $f, g \in A$ $\quad f = g$ on V $\iff \exists h \in I(V): f = g + h$

$\iff f \equiv g \mod I(V)$

$\iff f - g \equiv 0 \mod I(V)$

Thus the ring of polynomials on V is A modulo the equivalence relation $f \sim g \iff f - g \in I(V)$

This is exactly $k[V] = A/I(V)$.

Note $k[V]$ is a reduced domain of finite type over k.

Ex 3) a) $V = \mathbb{A}^n_k$ $\quad k[V] = k[T_1, \dots, T_n]/0 = k[T_1, \dots, T_n]$

b) $V = V(T_1 - T_2^2) \subset \mathbb{A}^2_k$ $\quad k[V] = k[T_1, T_2]/(T_1 - T_2^2) \simeq k[T_1]$

c) $V = V(T_1 - T_2^2, T_3, T_4) \subset \mathbb{A}^4_k$ $\quad k[V] = k[T_1, \dots, T_4]/(T_1 - T_2^2, T_3, T_4)$

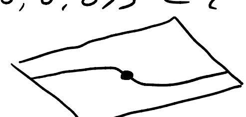
$\simeq k[T_1, T_2]/(T_1 - T_2^2)$

d) $V = \emptyset$ $\quad k[V] = k[T]/k[T] \simeq 0$

Def. Let $V \subset \mathbb{A}^n$ be an affine variety we endow it with the subspace topology inherited from $\mathbb{A}^n$.
An *affine subvariety* of V is a closed and irreducible subset of V.

In this language an affine variety $V \subset \mathbb{A}^n$ is an affine subvariety of $\mathbb{A}^n$.

Ex) $\{(0,0,0)\} \subset \{(t, t^3, 0) \mid t \in k\} \subset \{(t, t, 0) \mid t \in \mathbb{A}\} \subset \mathbb{A}^3_k$



Ex) Fix an affine variety V.

$$\left\{\begin{matrix}\text{closed subsets}\\ \text{of } V\end{matrix}\right\} \underset{V(\cdot)}{\overset{I(\cdot)}{\rightleftarrows}} \left\{\begin{matrix}\text{radical ideals}\\ \text{of } k[V]\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{affine subvarieties}\\ \text{of } V\end{matrix}\right\} \underset{V(\cdot)}{\overset{I(\cdot)}{\rightleftarrows}} \left\{\begin{matrix}\text{prime ideals}\\ \text{of } k[V]\end{matrix}\right\}$$

- $I(\cdot)$ and $V(\cdot)$ are inclusion reversing
- $W \subset V$ closed $\quad V(I(W)) = W$
- $\mathfrak{a} \subset k[V]$ ideal $\quad I(V(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}$

Now in the language of affine subvarieties we again have the following theorems

Thm (Basis thm) All ideals of $k[V]$ are finitely generated.

Thm (Nullstellensatz) For an affine variety V and k alg closed and $\mathfrak{a} \subset k[V]$ ideal $\quad I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

If k is algebraically closed, then we have the following inclusion reversing bijections

$$\left\{\begin{matrix}\text{affine subvarieties}\\ \text{of } V\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{prime ideals}\\ \text{of } k[V]\end{matrix}\right\}$$

$$\left\{\text{points of } V\right\} \longleftrightarrow \left\{\begin{matrix}\text{maximal ideals}\\ \text{of } k[V]\end{matrix}\right\}$$

Thus in order to understand varieties it suffices to understand the primes of quotients of polynomial rings (assuming k algebraically closed).

Morphisms

So far we have discussed our objects varieties, but arguably the more important consept is the maps between varieties. This focus is perfectly reasonable if we view maps of varieties as covers of some fixed space, e.g., all maps $V \to \mathbb{A}^1$ tell the story of varieties living over the line.

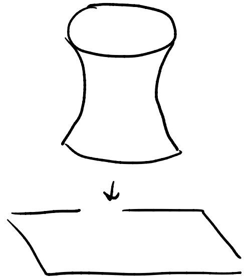
Def. Let $V \subset \mathbb{A}^n_k$ be an affine variety. A polynomial functional is a map $f: V \to \mathbb{A}^1_k$ such that there exists a $P \in k[T_1, \dots, T_n]$: $f(x) = P(x)$ for all $x \in V$

Note f is not uniquely represented.
Let $P, Q \in k[T_1, \dots, T_n]$: $P(x) = Q(x)$ $\forall x \in V$, then $P \equiv Q \mod I(V)$.

Def. Let $V \subset \mathbb{A}^n_k$ and $W \subset \mathbb{A}^m_k$ be two affine varieties a map $\varphi: V \to W$ is a polynomial morphism provided $\exists$ polynomial functionals $f_1, \dots, f_m$ on V :
$\varphi = (f_1, \dots, f_m)$

eg/ a) $\varphi : V(T_1^2 + T_2^2 + T_3^2 - 1) \longrightarrow \mathbb{A}^2$

$$(x_1, x_2, x_3) \longrightarrow (x_1, x_2)$$



b) Noether normalization

Given an affine variety V there exists an integer n and a polynomial morphism $\varphi : V \to \mathbb{A}^n$ such that φ has finite fibres.

c) let $\operatorname{char} k = p$. $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ $\quad x \mapsto x^p$

this is a bijective polynomial morphism

d) Given $V \xrightarrow{\varphi} W$, $\varphi^* : k[W] \to k[V]$

$$g \longmapsto g \circ \varphi$$

$g \circ \varphi(x) = 0$ since $\varphi(x) \in W$ for $x \in V$

$\therefore$ φ^* is well defined k-alg

Thm There is an equivalence of categories

$$\left\{ \begin{matrix} \text{affine varieties} \\ \text{over } k \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{reduced, finitely generated} \\ k\text{-algebras which are domains} \end{matrix} \right\}^{op}$$

$$\begin{array}{ccc} V & & k[V] \\ \varphi \downarrow & \longmapsto & \uparrow \varphi^{\#} \\ W & & k[W] \end{array}$$

This thm leads to the foundations of algebraic geometry.

Namely that any property of affine varieties can be answered in commutative algebra.

Affine Schemes

The equivalence of affine varieties and reduced finitely generated k-alg which are domains should lead to the following questions: what new geometry do we obtain by dropping the qualifiers on the algebra side?

A look to the future equivalences:

Geometry	$\longleftrightarrow$	Algebra
• points		• fields
• Affine varieties		• reduced, fg k-alg, domain
• algebraic sets		• reduced, fg k-alg
• ???		• commutative ring

When we drop all qualifiers and are left with a commutative ring A, the geometric object which corresponds to A is called an affine scheme.

Def. The category of affine schemes Aff is the opposite category of CRing.

Unfortunately this definition is useless until we prove what the "oposite" of a ring homomorphism even is.

From now on ring will mean commutative ring with unity and ring maps are required to preserve 1.

Def. Let A be a ring. $\operatorname{Spec} A = \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ prime ideal}\}$

Our end goal is to construct the opposite of A, which some how should be geometrical. Now taking inspiration from prime ideals of $k[V]$ and subvarieties of V, If we replace $k[V]$ w/ A, then our niave hope is that primes of A some how hold geometric properties. In the $k[V]$ case an ideal $\mathfrak{a}$ gave a closed set via

$$V(\mathfrak{a}) = \{x \in V \mid f(x) = 0 \ \forall f \in \mathfrak{a}\}.$$
$$= \{x \in V \mid \mathfrak{a} \subset \mathfrak{p}_x\}$$

where $\mathfrak{p}_x$ is max ideal corresponding to x.

Def. let $\mathfrak{a} \subset A$ be an ideal. $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subset \mathfrak{p}\}$

Prop
a) $V(0) = A \qquad V(A) = \emptyset$
b) let $\mathfrak{a}, \mathfrak{b} \subset A$ ideals $\quad V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$
c) let $\{\mathfrak{a}_i\}_{i \in I}$ be a family of ideals of A, then
$$\bigcap_{i \in I} V(\mathfrak{a}_i) = V\left(\sum_{i \in I} \mathfrak{a}_i\right)$$

Hence $V(\mathfrak{a})$ satisfy the axioms for closed sets on $\operatorname{Spec} A$. We call the topology on $\operatorname{Spec} A$ the Zariski topology.

Def. Let $f \in A$, $D(f) := \operatorname{Spec} A \setminus V(fA)$ is called a distinguished open in $\operatorname{Spec} A$.

Prop $\mathcal{B} = \{D(f) \mid f \in A\}$ is a basis for $\operatorname{Spec} A$.

Sheaves

We will need to define in order presheaves, sheaves, and then ringed spaces before we can pin down the opposite of a ring.

In complex geometry, to study a complex manifold M we need to understand holomorphic functions $f: U \to \mathbb{C}$ where $U \subset M$ is open. To consider all such functions at once would be to have a functor $\mathcal{O}_M : \{U \subset M \text{ open}\}^{op} \to \mathrm{CRing}$

$$\begin{array}{ccc} U & & \mathcal{O}_M(U) = \{f: U \to \mathbb{C} \text{ holo}\} \\ \uparrow & \longmapsto & \downarrow (-)|_W \\ W & & \mathcal{O}_M(W) = \{f: W \to \mathbb{C} \text{ holo}\} \end{array}$$

Following this idea to understand the geometry of $\mathrm{Spec}\, A$ we will study functions defined on open subsets $U \subset \mathrm{Spec}\, A$. These are called regular functions.

Def. A functor $\mathcal{O}: \mathcal{C}^{op} \to \mathcal{D}$ is called a **$\mathcal{D}$-valued presheaf**

$$\begin{array}{ccc} U & & \mathcal{O}(U) \\ \uparrow & \longmapsto & \downarrow \quad (-)|_W \text{ or } \mathrm{res}_{UW}(-) \\ W & & \mathcal{O}(W) \end{array}$$

Elements $s \in \mathcal{O}(U)$ are called **sections** of U

eg) • $\mathcal{O}_M$ is a ring valued presheaf

Rmk: $\mathcal{D}$ is often assumed to be locally small

I will basically only consider $\mathcal{D} \in \{\mathrm{Ab}, \mathrm{CRing}, A\text{-mod}\}$ unless otherwise stated.

Def. Given a top space X a **presheaf on X is** a presheaf $\mathcal{O}: \mathrm{op}(X)^{op} \to \mathcal{D}$

A key property about $\mathcal{O}_M$ is that for $\mathcal{U} \subset M$ open and $\{\mathcal{U}_i\}_{i \in I}$ open cover of $\mathcal{U}$. If $f_i : \mathcal{U}_i \to \mathbb{C} \in \mathcal{O}_M(\mathcal{U}_i)$ st. $f_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = f_j|_{\mathcal{U}_i \cap \mathcal{U}_j} \Rightarrow \exists!\ f : \mathcal{U} \to \mathbb{C} \in \mathcal{O}_M(\mathcal{U}) : f|_{\mathcal{U}_i} = f_i$. This property tells that holomorphic functions are uniquely determined by there values locally. Any civilized geometric theory ought to strive for this property.

The condition that f is unique is equivalent to if $f_i = 0$, then $f = 0$. Suppose $g \in \mathcal{O}_M(\mathcal{U}) : g|_{\mathcal{U}_i} = f_i \ \forall i$
$\therefore (f-g)|_{\mathcal{U}_i} = f|_{\mathcal{U}_i} - g|_{\mathcal{U}_i} = f_i - f_i = 0 \quad \forall i$
$\therefore f - g = 0 \quad \therefore f = g$. The converse is also clear.

Note that we have the following maps

$$\mathcal{O}_M(\mathcal{U}) \xrightarrow{\iota} \prod_i \mathcal{O}_M(\mathcal{U}_i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{i,j} \mathcal{O}_M(\mathcal{U}_i \cap \mathcal{U}_j)$$

$$f \longmapsto (f|_{\mathcal{U}_i})_i$$

$$(f_i)_i \longmapsto (f_i|_{\mathcal{U}_i \cap \mathcal{U}_j})_{ij}$$

$$ \longmapsto (f_j|_{\mathcal{U}_i \cap \mathcal{U}_j})_{ij}$$

The existence of gluing sections $\Leftrightarrow \ker(\alpha - \beta) = \operatorname{im} \iota$
uniqueness of gluing sections $\Leftrightarrow \iota$ monomorphism
Hence $\mathcal{O}_M(\mathcal{U}) = \operatorname{eq}\left(\prod_i \mathcal{O}_M(\mathcal{U}_i) \rightrightarrows \prod_{ij} \mathcal{O}_M(\mathcal{U}_i \cap \mathcal{U}_j)\right)$

Def. Given a top. sp. X a **sheaf on X** is a presheaf $\mathcal{O}$ on X such that $\forall\, \mathcal{U} \subset X$ open and for any open cover $\{\mathcal{U}_i\}_{i \in I}$ of $\mathcal{U}$ we have

$$\mathcal{O}(\mathcal{U}) = \operatorname{eq}\left(\prod_i \mathcal{O}(\mathcal{U}_i) \rightrightarrows \prod_{i,j} \mathcal{O}(\mathcal{U}_i \cap \mathcal{U}_j)\right).$$

So far to construct a presheaf we would need to define $\mathcal{O}(\mathcal{U})$ for all open $\mathcal{U}$, to simplify we only need to specify $\mathcal{O}(\mathcal{U})$ for basis elements $\mathcal{U}$.

Def. Let $\mathcal{B}$ be a basis for a top. sp X a $\mathcal{B}$ – (pre)sheaf is a (pre)sheaf $\mathcal{O}^{\mathcal{B}} : \mathcal{B}^{op} \to \mathcal{D}$.

Prop Given a basis $\mathcal{B}$ on a top. sp. X and a $\mathcal{B}$ – (pre)sheaf there exists a unique (pre)sheaf on X (up to isomorphism) st. $\mathcal{O}^{\mathcal{B}}(B) = \mathcal{O}(B)$ $\forall B \in \mathcal{B}$.

e.g.) Germs of holomorphic functions

For M a complex manifold the germ of holomorphic functions about $x \in M$ is defined to be

$$\mathcal{O}_{M,p} := \{ (f, \mathcal{U}) \mid x \in \mathcal{U} \subset M \text{ open},\ f \in \mathcal{O}_M(\mathcal{U}) \} / \sim$$

$$(f, \mathcal{U}) \sim (g, V) \iff f|_{\mathcal{U} \cap V} = g|_{\mathcal{U} \cap V}$$

i.e., $\mathcal{O}_{M,x}$ contains all functions that are holomorphic near x.

We have the following canonical maps for nbds $\mathcal{U} \ni x$

$$\mathcal{O}_M(\mathcal{U}) \longrightarrow \mathcal{O}_{M,x} \qquad f \mapsto (f, \mathcal{U})$$

If $x \in \mathcal{W} \subset \mathcal{U}$, then

$$\begin{tikzcd}
\mathcal{O}_M(\mathcal{U}) \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\#"] & \mathcal{O}_M(\mathcal{W}) \arrow[dl] \\
& \mathcal{O}_{M,x} &
\end{tikzcd}$$

Since if $x \in \mathcal{W}$ and $x \in \overset{\text{open}}{\mathcal{U}}$, then $x \in \mathcal{W} \cap \mathcal{U}$ open then $\exists\ \mathcal{O}_M(\mathcal{W}) \to \mathcal{O}_M(\mathcal{U} \cap \mathcal{W})$ and $\exists\ \mathcal{O}_M(\mathcal{U}) \to \mathcal{O}_M(\mathcal{U} \cap \mathcal{W})$

ie $\{ \mathcal{O}_M(\mathcal{U}) \mid x \in \mathcal{U} \}$ is a directed set.

Thus $\varinjlim_{\mathcal{U}\ni x} \mathcal{O}_M(\mathcal{U})$ exists and we have
$\varinjlim_{\mathcal{U}\ni x} \mathcal{O}_M(\mathcal{U}) \longrightarrow \mathcal{O}_{M,x}$. This turns out to be an isomorphism.

Def. Given a presheaf $\mathcal{O}$ on X and $x \in X$, we define the **stalk** of $\mathcal{O}$ at x as $\mathcal{O}_x = \varinjlim_{\mathcal{U}\ni x} \mathcal{O}(\mathcal{U})$

Def. Given (pre)sheafs $\mathcal{F}$ and $\mathcal{G}$ on X we define a **morphism** $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ is a natural transformation
ie if $\mathcal{W} \subset \mathcal{U}$ are open in X, then

$$\begin{tikzcd}
\mathcal{F}(\mathcal{U}) \arrow[r, "\alpha(\mathcal{U})"] \arrow[d, "(-)|_{\mathcal{W}}"'] \arrow[dr, phantom, "\circledast"] & \mathcal{G}(\mathcal{U}) \arrow[d, "(-)|_{\mathcal{W}}"] \\
\mathcal{F}(\mathcal{W}) \arrow[r, "\alpha(\mathcal{W})"'] & \mathcal{G}(\mathcal{W})
\end{tikzcd}$$

e.g.) Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X and $x \in X$
let $x \in \mathcal{W} \subset \mathcal{U}$ be opens in X

$$\begin{tikzcd}
\mathcal{F}(\mathcal{U}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circledast"] & \mathcal{G}(\mathcal{U}) \arrow[d] \\
\mathcal{F}(\mathcal{W}) \arrow[r] & \mathcal{G}(\mathcal{W})
\end{tikzcd} \rightsquigarrow \exists\ \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x$$

by universal prop of direct limits.

Prop $\alpha : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves

α mono $\iff$ α_x mono $\forall x \in X$

α epi $\iff$ α_x epi $\forall x \in X$

α iso $\iff$ α_x iso $\forall x \in X$

e.g.) Let $\mathcal{O}_{\mathbb{C}}$ be the sheaf of holo fns on $\mathbb{C}$
and $\mathcal{O}_{\mathbb{C}}^{\times}$ be the sheaf of holo fns on $\mathbb{C}$ which are nonvanishing on their domain.

Let $\exp : \mathcal{O}_{\mathbb{C}} \to \mathcal{O}_{\mathbb{C}}^{\times}$ be defined by

$$\exp(\mathcal{U}) : \mathcal{O}_{\mathbb{C}}(\mathcal{U}) \to \mathcal{O}_{\mathbb{C}}^{\times}(\mathcal{U})$$
$$f \mapsto e^{f}$$

If $\mathcal{W} \subset \mathcal{U}$,

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{C}}(\mathcal{U}) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{C}}^{\times}(\mathcal{U}) \arrow[d] \\
\mathcal{O}_{\mathbb{C}}(\mathcal{W}) \arrow[r] & \mathcal{O}_{\mathbb{C}}^{\times}(\mathcal{W})
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\#"] & e^{f} \arrow[d, mapsto] \\
f|_{\mathcal{W}} \arrow[r, mapsto] & e^{f|_{\mathcal{W}}}
\end{tikzcd}$$

$$e^{f|_{\mathcal{W}}} = e^{f}|_{\mathcal{W}}$$

Let $x \in \mathbb{C}$, $\exp_x : \mathcal{O}_{\mathbb{C},x} \to \mathcal{O}_{\mathbb{C},x}^{\times}$

$$(f, \mathcal{U}) \mapsto (e^{f}, \mathcal{U})$$

Let $(g, \mathcal{U}) \in \mathcal{O}_{\mathbb{C},x}^{\times}$ ie $g : \mathcal{U} \to \mathbb{C}$ holo and $x \in \mathcal{U}$ and $g(y) \neq 0 \ \forall y \in \mathcal{U}$.

$\exists \varepsilon > 0 : B = B_{\varepsilon}(x) \subset \mathcal{U} \quad \therefore \quad (g|_B, B) = (g, \mathcal{U}) \in \mathcal{O}_{\mathbb{C},x}^{\times}$

Since g is nowhere vanishing on a simply connected domain $\exists\, G : B \to \mathbb{C}$ holo : $e^{G} = g$ on B.

ie $\exp_x(G, B) = (g|_B, B) = (g, \mathcal{U})$.

$\therefore$ $\exp_x$ is epi $\therefore$ $\exp$ is epi.

Note that $\mathbb{C}^{\times} \subset \mathbb{C}$ open and

$$\mathcal{O}_{\mathbb{C}}(\mathbb{C}^{\times}) \longrightarrow \underset{\substack{\Psi \\ 1/z}}{\mathcal{O}_{\mathbb{C}}^{\times}(\mathbb{C}^{\times})}$$

but $1/z$ does not have a log on $\mathbb{C}^{\times}$.

$\therefore$ $\exp(\mathbb{C}^{\times})$ is not mono.

e.g.1 Sheafification Here we will describe a process in which we take $\mathcal{O} \in \mathrm{Psh}\, X$ and force it to be a sheaf in some free sense.

Let $U \subset X$ be open and $\mathcal{U} = \{U_i\}$ be an open cover of U. Define

$$\mathcal{O}_{\mathcal{U}}(U) = H^0(\mathcal{U}, \mathcal{O}) := eq\left(\prod_i \mathcal{O}(U_i) \rightrightarrows \prod_{i,j} \mathcal{O}(U_i \cap U_j) \right)$$

This should be reminiscent of the sheaf condition so we may think our sheaf of sections over U will simply be this equalizer. Unfortunatly $H^0(\mathcal{U}, \mathcal{O})$ depends on our cover $\mathcal{U}$. To remove this dependency we will consider all covers of U at once using a direct limit.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ be open covers of U. We say $\mathcal{V}$ is a refinement of $\mathcal{U}$ if there exists a map $\sigma : J \to I$: $V_j \subset U_{\sigma j}$ $\forall j \in J$. In this case we say $\sigma : \mathcal{V} \to \mathcal{U}$.
If $\mathcal{U}, \mathcal{V}$ are two open coverings of U, then let $\mathcal{W} = \{ W_{ij} = U_i \cap V_j \}_{(i,j) \in I \times J}$.
Let

$$\sigma : I \times J \to I, \quad i,j \mapsto i \qquad \text{and} \qquad \tau : I \times J \to J, \quad i,j \mapsto j$$

then $W_{ij} \subset U_i$ and $W_{ij} \subset V_j$.
$\therefore$ $\mathcal{W}$ is a refinement of $\mathcal{U}$ and $\mathcal{V}$.

Let $\mathcal{V} < \mathcal{U}$ and $W \subset X$ open. $\forall k\ \exists i\ V_k \subset U_i$

$$\begin{tikzcd} \mathcal{O}_{\mathcal{U}}(W) \arrow[r] \arrow[d, dashed] & \prod_i \mathcal{O}(U_i \cap W) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{ij} \mathcal{O}(U_i \cap U_j \cap W) \arrow[d] \\ \mathcal{O}_{\mathcal{V}}(W) \arrow[r] & \prod_k \mathcal{O}(V_k \cap W) \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, phantom, "\#"] & \prod_{ij} \mathcal{O}(V_k \cap V_\ell \cap W) \end{tikzcd}$$

where $\mathcal{O}(U_i \cap W) \longrightarrow \prod_k \mathcal{O}(V_k \cap W)$ where $V_k \subset U_i$.
$f \longmapsto (f|_{V_k \cap W})_k$

Suppose $\mathcal{V}$ and $\mathcal{U}$ are two open covers of X
Let $\mathcal{W} = \{V_k \cap U_i\}_{ik}$, then $W < V$ and $W < U$
$\therefore \{\mathcal{V} \mid \mathcal{V}$ open cover of $X\}$ is a direct system.

$\mathcal{O}^{sh}(W) := \operatorname{colim}_{\mathcal{V}} \mathcal{O}_{\mathcal{V}}(W)$ for $W \subset X$ open.

If $W \subset U$ opens

$$\begin{tikzcd} \mathcal{O}^{sh}(U) = \operatorname{colim}_{\mathcal{V}} \mathcal{O}_{\mathcal{V}}(U) & \mathcal{O}_{\mathcal{V}}(U) \arrow[l] \\ \mathcal{O}^{sh}(W) = \operatorname{colim}_{\mathcal{V}} \mathcal{O}_{\mathcal{V}}(W) \arrow[u, dashed] & \mathcal{O}_{\mathcal{V}}(W) \arrow[l] \arrow[u] \end{tikzcd}$$

So $\mathcal{O}^{sh}$ is a presheaf.

Let $U \subset X$ open and $\{U_i\}$ an open cover of U.

$$\mathcal{O}^{sh}(U) \longrightarrow \prod_i \mathcal{O}^{sh}(U_i) \rightrightarrows \prod_{ij} \mathcal{O}^{sh}(U_i \cap U_j)$$

is exact. Hence $\mathcal{O}^{sh}$ is a sheaf.

Prop Let A be a ring, then $\{D(f) \mid f \in A\}$ is a basis for $\operatorname{Spec} A$.

e.g.) Let A be a ring we define the sheaf of regular functions on $\operatorname{Spec} A$ by

$$\mathcal{O}_A(D(f)) = A[1/f] \quad \text{for all } f \in A$$

Suppose $D(f) \subset D(g)$, then $V(gA) \subset V(fA)$

One may show this implies $\exists n \in \mathbb{N}\ \exists a \in A : f^n = ga$

Hence in $A[1/f]$, $1 = g \cdot (a/f^n)$.

This implies in $A[1/f]$ g is a unit.

By the universal prop of localization we obtain a map $A[1/g] \to A[1/f]$.

If $D(f) = D(g)$, then one may show $A[1/f] = A[1/g]$

$\therefore$ We have the data for a $\mathcal{B}$-presheaf on $\operatorname{Spec} A$.

Thm $\mathcal{O}_A$ is a sheaf.

e.g.) Push forward and Pull back

Let $f : X \to Y$ be a continuous map of top. sp.

Let $\mathcal{O}_X$ be a (pre)sheaf on X

$\mathcal{O}_Y$ be a (pre)sheaf on Y

Push forward $(f_* \mathcal{O}_X)(V) := \mathcal{O}_X(f^{-1}(V))$ $\quad V \subset Y$ open

Pull back $(f_p^{-1} \mathcal{O}_Y)(U) := \varinjlim_{V \supset U} \mathcal{O}_Y(V)$

if $\mathcal{O}_X$ is a sheaf so is $f_* \mathcal{O}_X$

if $\mathcal{O}_Y$ is a sheaf, $f_p^{-1} \mathcal{O}_Y$ is not always a sheaf.

so we let $f^{-1}\mathcal{O}_Y = (f_p^{-1} \mathcal{O}_Y)^{sh}$.

Ringed Spaces and Schemes.

Maps defined on geometric objects are the best tool for studying the behaviour of shapes beyond our imagination. Examples of this pairing would be top. sp. and continuous maps, manifolds and smooth functions, or Riemann surfaces and holomorphic functions.

Def. A **ringed space** is a pair $(X, \mathcal{O})$ where X is a top. sp and $\mathcal{O}$ is a ring valued sheaf on X.

A **locally ringed space** is a ringed space $(X, \mathcal{O})$ such that $\mathcal{O}_x$ is a local ring $\forall x \in X$.

Def. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces the pair $(f, f^{\#})$ is a **morphism** from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ if $f: X \to Y$ is continuous and $f^{\#}: \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves

Furthermore, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces we say $(f, f^{\#})$ is a **morphism of locally ringed spaces** if $f^{\#}_x: \mathcal{O}_{Y,x} \to \mathcal{O}_x$ is a local homomorphism.

Prop $(\operatorname{Spec} A, \mathcal{O}_A)$ is a locally ringed space

Thm $\left\{ \begin{array}{l} (\operatorname{Spec} A, \mathcal{O}_A) \\ \text{morphisms of} \\ \text{locally ringed spaces} \end{array} \right\} \underset{\text{equivalent}}{\longleftrightarrow} \mathrm{CRing}^{op}$

Hence $(\operatorname{Spec} A, \mathcal{O}_A)$ are the mysterious affine schemes.

e.g.) Let $\mathcal{O}$ be a presheaf on X and $\iota : U \hookrightarrow X$ open
$\mathcal{O}|_U := \iota^{-1}\mathcal{O}$
if $V \subset U$ open, then $V \subset X$ open $\therefore$ $\mathcal{O}|_U(V) = \mathcal{O}(V)$.

e.g.) • A manifold is a ringed space (X, C^∞)
such that $\forall x \in X$ $\exists$ nbd U of x:
$(U, C^\infty|_U) \simeq (B_\varepsilon(0) \subset k^n, C^\infty(B_\varepsilon(0), k))$
as ringed spaces ($k \in \{\mathbb{R}, \mathbb{C}\}$).
We say (X, C^∞) is locally Euclidean.

Def. A scheme is a locally ringed space $(X, \mathcal{O})$ st.
$\forall x \in X$ $\exists$ nbd U of x $\exists$ $A \in \mathrm{CRing}$ st.
$(U, \mathcal{O}|_U) \simeq (\operatorname{Spec} A, \mathcal{O}_A)$.
"A scheme is a locally ringed space that is locally affine"