Lesson 28 : Section 4.7

L'Hôpital's Rule

If

$$\lim_{x \to a} \frac{f(x)}{g(x)} \text{ "="} \frac{0}{0} \text{ or } \frac{\infty}{\infty},$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Caution:

- To apply L'Hôpital's Rule, take the derivative of $f(x)$ and $g(x)$ separately, NOT the quotient rule.
- L'Hôpital's Rule only applies to the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

Manipulating Other Indeterminate Forms to Take Advantage of L'Hôpital's Rule

- If $\lim_{x \to a} f(x) \cdot g(x)$ "=" $0 \cdot \infty$, then $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} \dfrac{f(x)}{\frac{1}{g(x)}}$.
- If $\lim_{x \to a} (f(x) - g(x))$ "=" $\infty - \infty$, use algebra to get an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. (Often, this means get a common denominator)
- If $\lim_{x \to a} f(x)^{g(x)}$ "=" $1^{\infty}, 0^{0}, \infty^{0}$, compute $L = \underbrace{\lim_{x \to a} g(x) \cdot \ln(f(x))}_{\text{form } 0 \cdot \infty}$.
  - If L = number, then $\lim_{x \to a} f(x)^{g(x)} = e^{L}$
  - If $L = \infty$, then $\lim_{x \to a} f(x)^{g(x)} = \infty$
  - If $L = -\infty$, then $\lim_{x \to a} f(x)^{g(x)} = 0$

Growth Rates as $x \to \infty$

- If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then <u>$f$ grows faster than g</u> if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

- Writing "$g \ll f$" to mean "f grows faster than g," we have

$$(\ln(x))^q \ll x^p \ll x^p(\ln(x))^r \ll x^{p+s} \ll b^x \ll x^x$$

for positive numbers, q, p, r, s and $b > 1$.

<u>Example:</u> $\lim_{x\to\infty} \frac{2^x}{x^{2020}} = \infty$

Since $x^{2020} \ll 2^x$. (Or apply L'Hôpital's Rule 2020 times!)

Practice Problems

1. If $a > 0$ is a constant, compute

$$\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}.$$

2. Compute

$$\lim_{x\to 0} \left(e^x + 5x\right)^{1/x}$$