MA 16100 | Midterm Exam 2 | Summer 2019

Name: Key

ID number: ______________________

Instructions:

1. This is a one-hour exam. If you continue to work on this exam after time is called it will be considered cheating and you will receive a zero.
2. There are 11 problems on this exam.
3. No books, notes, or calculators are allowed. Only a writing utensil, eraser, and water are allowed on your desk with the exam.
4. Turn off your cell phone.
5. Circle ***one and only one choice*** for each multiple-choice problem. Showing your work is NOT necessary for multiple-choice problems and no partial credit will be given for multiple-choice problems.
6. Legibly show all relevant work on non-multiple-choice problems. Partial credit will be given for steps leading to the correct solutions.
Little or no work with a correct answer will receive little or no credit.

I agree to abide by the instructions above and have read and understood the above statement regarding academic integrity:

Signature: ______________________

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1. (8 points) Find the equation of the line tangent to the curve, $\sin(x - y) + x^3y = y + 14$, at the point $(2, 2)$.

Point: $(2,2)$

Slope: $y'(2)$

$$\frac{d}{dx}\left[\sin(x-y) + x^3 y\right] = \frac{d}{dx}\left[y + 14\right]$$

✓ for each side

$$\cos(x-y)\cdot(1-y') + 3x^2y + x^3y' = y'$$

$$\left[-\cos(x-y) + x^3 - 1\right]y' = -\cos(x-y) - 3x^2y$$

$$\left(-\cos(0) + 2^3 - 1\right)y' = -\cos(0) - 3\cdot 2^3$$

Plug in (2,2)

$$(-1 + 8 - 1)\,y' = -1 - 24$$

$$y' = \frac{-25}{6}$$

$$\boxed{y - 2 = \frac{-25}{6}(x - 2)}$$

$$y = -\frac{25}{6}x + \frac{50}{6} + 2$$

$$+\frac{31}{3}$$

2. (10 points) Evaluate $y''(1)$ where $y = e^x + x^e$.

A. 0
B. 1
C. 2
D. e
E. $\boxed{e^2}$

$y' = e^x + e\,x^{e-1}$

$y'' = e^x + e \cdot (e-1)\,x^{e-2}$

$y'' = e + e^2 - e = e^2$

3. (8 points) Evaluate the derivative of the *inverse* of $f(x) = x^3 - x + 1$ at the point $(1, -1)$.

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$

$(f^{-1})'(1) = \frac{1}{f'(-1)}$

$= \frac{1}{2}$

$f'(x) = 3x^2 - 1$

$f'(-1) = 3 - 1 = 2$

$\boxed{1/2}$

4. (8 points) The graph of $y = x^{\ln(x)+1}$, $x > 0$, has exactly one horizontal tangent line. Find the equation of the horizontal tangent line.

Use logarithmic differentiation ✓

Horizontal tangent line : $y' = 0$ ✓

$$\ln y = (\ln x + 1) \cdot \ln x \checkmark$$

$$\frac{d}{dx} \ln y = \frac{d}{dx}\left[(\ln x + 1)\ln x\right]$$

$$\frac{y'}{y} \checkmark = \frac{1}{x}\ln x + \frac{1}{x}(\ln x + 1) \checkmark$$

or y

$$y' = \left(x^{\ln x + 1}\right) \cdot \frac{1}{x}(2\ln x + 1) \checkmark$$

6

Equal to zero when $2\ln x + 1 = 0$ ✓

7 points

$\ln x = -1/2$ ✓

$x = e^{-1/2}$ ✓

$y = (e^{-1/2})^{\ln(e^{-1/2})+1}$ ✓ ← full credit

$y = (e^{-1/2})^{1/2}$

$$\boxed{y = e^{-1/4} \checkmark}$$

5. Evaluate the derivatives of the following functions:

i. (4 points) Evaluate the derivative of $g(z) = \tan^{-1}(1/z)$ at $z = 1$.

$g'(z) = \frac{1}{1+(\frac{1}{z})^2} \cdot \left(-\frac{1}{z^2}\right) = \frac{-1}{1+z^2}$

$g'(1) = \frac{1}{1+1} \cdot (-1)$

$= -\frac{1}{2}$

$\boxed{-1/2}$

ii. (4 points) Evaluate the derivative of $h(z) = \dfrac{1}{\cos^{-1}(z)}$ at $z = 0$.

$h(z) = (\cos^{-1}(z))^{-1}$

$h'(z) = -(\cos^{-1}(z))^{-2} \cdot \frac{-1}{\sqrt{1-z^2}}$

$h'(0) = -(\cos^{-1}(0))^{-2} \cdot \frac{-1}{1}$

$= -\left(\frac{\pi}{2}\right)^{-2} \cdot -1$

$h'(z) = \frac{+1/\sqrt{1-z^2}}{(\cos^{-1}(z))^2}$

$h'(0) = \frac{+1}{(\cos^{-1}(0))^2}$

$h'(0) = \frac{1}{(\frac{\pi}{2})^2}$

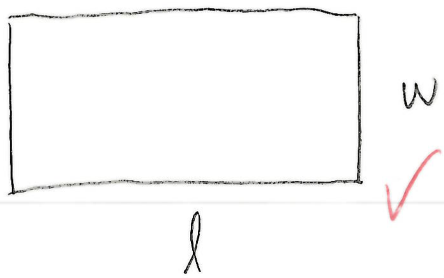
$\boxed{4/\pi^2}$

6. (8 points) The length of a rectangle is increasing at a rate of 7 cm/s and its width is increasing at a rate of 3 cm/s.

When the length is 12 cm and the width is 8 cm, how fast is the area of the rectangle increasing?

3-5 pts for set up

Diagram: 2pts



Relation: Area $A = lw$

Rates: $\frac{dl}{dt} = 7 \frac{cm}{s}$; $\frac{dw}{dt} = 3 \frac{cm}{s}$

Rate Relation: $A' = w \frac{dl}{dt} + l \frac{dw}{dt} = 7w + 3l$

when $l = 12$ cm and $w = 8$ cm,

$A' = 7 \cdot 8 + 3 \cdot 12 = 56 + 36$

$A' = 92$ cm/s

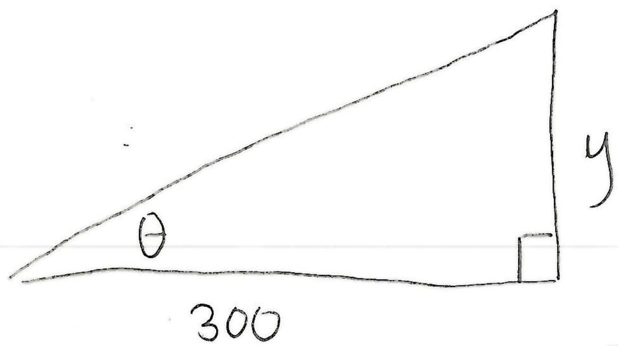
$\boxed{92}$

cm/s

7. (9 points) An observer stands 300 ft from the launch site of a hot-air balloon at an elevation equal to the elevation of the launch site. The balloon is launched vertically and maintains a constant upward velocity of 20 ft/s.

What is the rate of change of the angle of elevation of the balloon when it is 400 ft from the ground?

Diagram: ✓
Labels ✓



Relation: $\tan\theta = \frac{y}{300}$ or $\theta = \tan^{-1}\left(\frac{y}{300}\right)$ ✓

Rates: $\frac{dy}{dt} = 20$ ft/s ✓

Rate Relation: ✓ $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{300}\frac{dy}{dt}$ or $\frac{d\theta}{dt} = \frac{1}{1+(y/300)^2}\cdot\frac{dy}{dt}\cdot\frac{1}{300}$

Plug in: ✓ $y = 400$ (then $z = 500$ ✓)

$\boxed{\left(\frac{500}{300}\right)^2 \frac{d\theta}{dt} = \frac{1}{300}\cdot 20}$ ✓ or $\boxed{\frac{d\theta}{dt} = \frac{1}{1+(400/300)^2}\cdot 20 \cdot \frac{1}{300}}$ ✓

$= \frac{20}{1+16/9} = 20 \cdot \frac{9}{25}\cdot\frac{1}{300}$

$\frac{d\theta}{dt} = \frac{20}{300}\cdot\left(\frac{3}{5}\right)^2$

$= \frac{20\cdot 9}{300\cdot 25}$

$= \frac{60}{2500}$

$= \frac{.6}{25.0}$

$= \frac{3}{125}$

$\boxed{3/125}$ ✓

rad/s

8. (8 points) Determine the value(s) of the absolute extreme values of $f(t) = \frac{3t}{t^2+1}$ on the interval $[-2, 0]$.

f is continuous on a closed interval ∴ it must have an absolute maximum and minimum

Find critical points:

$$f'(t) = \frac{3(t^2+1) - 3t \cdot 2t}{(t^2+1)^2}$$

$$= \frac{3t^2+3-6t^2}{(t^2+1)^2}$$

$$= \frac{-3(t^2-1)}{(t^2+1)^2}$$

$t=-1$ and $\cancel{t=1}$

Evaluate at critical points and endpoints

$f(-2) = \frac{-6}{5} = -1.2$

$f(-1) = \frac{-3}{2} = -1.5$

$f(0) = 0$

Absolute minimum value:

$\boxed{-1.5}$

Absolute maximum value:

$\boxed{0}$

9. (10 points) Which of the following functions satisfy the conditions of Rolle's Theorem on the interval $[-1, 1]$?

X $f(x) = 1 - x^{2/3}$ not differentiable at 0

✓ $g(x) = x^3 - 2x^2 - x + 2$ $g(-1) = g(1)$

X $h(x) = \cos\left(\frac{\pi}{4}(x+1)\right)$ $h(-1) \neq h(1)$

Rolle's Theorem applies to:

A. both f and g

B. both g and h

C. g only

D. h only

E. All three

10. i. (3 points) Fill in the blanks to complete the statement of the Mean Value Theorem:

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

ii. (6 points) The $f(x) = x + \frac{1}{x}$ on $[1, 3]$ satisfies the conditions of the Mean Value Theorem. Find the point(s), c, that is/are guaranteed to exist by the Mean Value Theorem.

Derivative: $f'(x) = 1 - \frac{1}{x^2}$

Slope of secant: $\frac{f(3) - f(1)}{3-1} = \frac{3 + \frac{1}{3} - (1+1)}{2}$

$= \frac{1 + \frac{1}{3}}{2} = \frac{2}{3}$

$f'(c) = 1 - \frac{1}{c^2} = \frac{2}{3}$

$\frac{1}{c^2} = \frac{1}{3}$

$c = \pm\sqrt{3}$

$\boxed{\sqrt{3}}$

11. Suppose that the derivative of a function h is given by:

$$h'(x) = x(x-1)^2(x-2)$$

i. (7 points) On what interval(s) is h increasing?

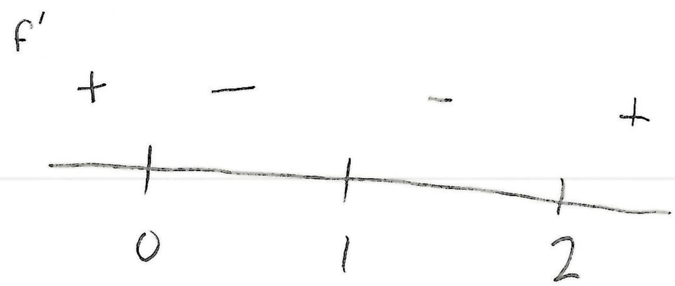
A. $(-\infty, 0)$

B. $(-\infty, 0)$ and $(2, \infty)$

C. $(0, 2)$

D. $(0, 1)$ and $(2, \infty)$

E. $(-\infty, 1)$



ii. (7 points) Where does h have a local maximum?

A. $x = 0$

B. $x = 2$

C. $x = 0$ and 1

D. $x = 0$ and 2

E. $x = 1$ and 2