MA 16020 - Quiz 12 - Spring 2016

Directions: Please show all the work leading to your answers. Having some correct work with an incorrect answer will earn you partial credit.

1. We want to construct a rectangular box with a volume of 24 cubic feet of minimal cost. The material for the top costs $12 per square foot, the four sides cost $8 per square foot, and the bottom costs $12 per square foot. To the nearest cent, what is the minimum cost for such a box?

   (a) Set up the cost as a function of length, width, and height
   (b) Reduce this to an equation in two of the variables
   (c) Find the minimum cost

NOTE: In the 10:30am section, I changed a few of the numbers (the top cost 9$ per square foot and the bottom cost $15 per square foot), but the final answer is the same

Solution

1.

(a) We first start by labelling the box as above. Then we can write down
a function that tells us the cost of constructing the box, given the dimensions x, y, and z. The cost will be: 12xy for the top, 12xy for the bottom, 2 · 8xz for the front and back sides together, and finally 2 · 8yz for the left and right sides. So the total cost is:

\[12xy + 12xy + 16xz + 16yz\]

(b) Now we can use the fact that the volume must be exactly 24 cubic feet so that \[xyz = 24\]. We’ll solve for \(z\) here and plug it into the equation above to get that the total cost as a function of \(x\) and \(y\) is:

\[
C(x, y) = 24xy + 384x^{-1} + 384y^{-1}
\]

(c) Now to find the minimum cost, we need to find critical points and possibly use the Second Derivative Test to figure out which one is a local minimum. To find the critical points we find where \(C_x = C_y = 0\):

\[
\begin{align*}
C_x &= 24y - 384x^{-2} \\
C_y &= 24x - 384y^{-2}
\end{align*}
\]

So we compute where \(24y - \frac{384}{x^2} = 0\) and \(24x - \frac{384}{y^2} = 0\). We can solve for \(y\) in the first equation to get that \(y = \frac{16}{x^2}\) and solve for \(x\) in the second to get \(x = \frac{16}{y^2}\). If we clear out the denominators, we get \(x^2y = 16 = xy^2\). Since neither \(x\) nor \(y\) is equal to 0 (because otherwise \(x^2y = 0 \neq 16\)), we can divide \(x^2y = xy^2\) by \(x\) and \(y\) to get that \(x = y\) and hence \(x^2y = x^3 = 16\) so that \(x = y = \sqrt[3]{16}\). Since there is only one critical point, it must be the local minimum (notice that it can’t be the local max because we can make \(x\) as small as we like, which means we can make \(x^{-1}\) and hence \(C(x, y)\) as big as we like). The last thing to do is evaluate \(C\) at this point:

\[
C(\sqrt[3]{16}, \sqrt[3]{16}) = 24(\sqrt[3]{16})^2 + \frac{384}{\sqrt[3]{16}} + \frac{384}{\sqrt[3]{16}} = 457.17
\]