Directions: Please show all the work leading to your answers. Having some correct work with an incorrect answer will earn you partial credit.

1. Find the minimum value of \( f(x, y) = x^2 + 4xy \) subject to the constraint \( x^2y = 108 \).

Solution

We need to use the Lagrange multiplier method:

Find \( x \), \( y \), and \( \lambda \) such that the following equations are true:

\[
\begin{align*}
2x + 4y &= \lambda (2xy) \\
4x &= \lambda (x^2) \\
x^2y &= 108 \\
\end{align*}
\]

The simplest equation to work with is \( 4x = \lambda x^2 \) since it only involves two variables. In order to divide by \( x \), we first need to check what happens when it is equal to zero. \( x = 0 \) implies that \( x^2y = 0 \neq 108 \). So we know that in order to satisfy all three equations, \( x \neq 0 \). So dividing by \( x \) gives us \( 4 = \lambda x \). At this point there are two ways of proceeding and both will give the correct solution:

Method 1: Plug \( 4 = \lambda x \) into the first equation

The right hand side of the first equation is \( 2\lambda xy \) and since \( \lambda = 4 \) it becomes \( 2(4)y = 8y \). So the first equation becomes

\[2x + 4y = 8y\]

so \( 2x = 4y \) and \( x = 2y \). Now we plug that into the third equation to get \( x^2y = (2y)^2y = 4y^3 = 108 \) and hence \( y^3 = 27 \) which means that \( y = 3 \). Since \( x = 2y \), \( x = 6 \).

Method 2: Plug \( 4 = \lambda x \) into the third equation

We have that \( x = \frac{4}{\lambda} \) so \( x^2y = (\frac{4}{\lambda})^2y = \frac{16}{\lambda^2}y = 108 \) so that \( y = \lambda^2 \frac{108}{16} = \frac{27}{4} \lambda^2 \). We can plug this and \( x = \frac{4}{\lambda} \) into the first equation to get that

\[
\begin{align*}
2 \frac{4}{\lambda} + 4 \cdot \frac{27}{4} \lambda^2 &= 2 \lambda^2 \frac{27}{4} \\
\frac{8}{\lambda} + 27 \lambda^2 &= 54 \lambda^2 \\
\frac{8}{\lambda} &= 27 \lambda^2 \\
\frac{8}{27} &= \lambda^3
\end{align*}
\]

so that \( \lambda = \frac{2}{3} \) and hence \( x = \frac{4}{\frac{2}{3}} = 4 \frac{3}{2} = 6 \) and \( y = \frac{27}{2} \frac{4}{9} = 3 \).

So the only point we get is \((6, 3)\) so we plug that into \( f \) to get \( f(6, 3) = 6^2 + 4 \cdot 6 \cdot 3 = 108 \).