18.100B
Fall 2012

Final exam review sheet

Most of the final will consist of problems from this list.

1. Prove that $\mathbb{R}^2$ is not the union of a countable family of lines.

2. Let $E \subset \mathbb{R}$ be uncountable.
   (a) Prove that E has uncountably many limit points.
   (b) Prove that there exists $x \in \mathbb{R}$ such that $E \cap (-\infty, x)$ and $E \cap (x, \infty)$ are both uncountable.

3. Problem 6 from page 99.

4. Let $f\colon [0,1] \times [0,1] \to \mathbb{R}$ be continuous, and let $g\colon [0,1] \to \mathbb{R}$ be defined by
$$g(x) = \max\{f(x,y)\colon y \in [0,1]\}.$$
Prove that g is continuous.

5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $f^{-1}(E) \subset \mathbb{R}^2$ is bounded whenever $E \subset \mathbb{R}$ is bounded. Prove that f attains either a maximum or a minimum value.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nonconstant, nondecreasing function. Prove that for any $R > 0$ there exist $a \in \mathbb{R}$ and $c > 0$ such that
$$f(a + x) - f(a - x) \geq cx,$$
for all $x \in [0, R]$.

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is surjective.

8. Problem 3 from page 114.

9. Let $f\colon (-1,1) \to \mathbb{R}$ be differentiable, and suppose $f(x)/x^2$ has a finite limit as $x \to 0$. Does it follow that $f''(0)$ exists?

10. Let $f\colon (0,\infty) \to (0,\infty)$ be twice differentiable with f'' bounded and con-

tinuous, and such that $f'(x) \leq 0$ for all $x > 0$. Prove that $\lim_{x \to \infty} f'(x) = 0$

11. Let $(a_n)_{n \in \mathbb{N}}$ be a bounded sequence of real numbers and let $f\colon [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} a_n, & x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

For which sequences a_n is f continuous at 0? For which is it differentiable at 0? For which is it Riemann-integrable on $[0,1]$?

12. Let $f\colon [0,1] \to \mathbb{R}$ be differentiable with continuous derivative, and with $f(0) = 0$. Prove that

$$\sup_{0 \leq x \leq 1} |f(x)| \leq \sqrt{\int_0^1 (f'(x))^2 dx.}$$

13. Problem 15 from page 141.

14. Let $f\colon [0,1] \to [0,1]$ be convex. Prove that the arclength of the graph of f is at most 3.

15. Problem 15 from page 168.

16. Let$(f_n)_{n \in \mathbb{N}}$ be a family of uniformly bounded differentiable functions on $[a,b]$. Suppose $(f_n')_{n \in \mathbb{N}}$ is uniformly bounded. Prove that $(f_n)_{n \in \mathbb{N}}$ has a convergent subsequnce.

17. Let K be a compact metric space, and let $(f_n)_{n \in \mathbb{N}}$ be a uniformly bounded equicontinuous family of functions $K \to \mathbb{R}$. For each $n \in \mathbb{N}$, define $g_n\colon K \to \mathbb{R}$ by

$$g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}.$$

Prove that the sequence $(g_n)_{n \in \mathbb{N}}$ converges uniformly.

18. Let $N \in \mathbb{N}$. Prove that if f is a continuous function $[0,1] \to \mathbb{R}$ satisfying

$$\int_0^1 x^n f(x) dx = 0$$

for all $n \in \mathbb{N}$ with $n \geq N$, then $f \equiv 0$.

19. For $n \in \mathbb{N}$, let $I_n = \displaystyle\int_0^\pi \sin^n x dx$.

(a) Evaluate I_n for each $n \in \mathbb{N}$.
(b) Prove that $(I_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence of positive terms.
(c) Evaluate the infinite product $\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdots$.

20. Let $R > 0$, and let $f \colon \mathbb{R} \to \mathbb{C}$ be integrable with $f(x) = 0$ for $|x| \geq R$. Let $\hat{f} \colon \mathbb{R} \to \mathbb{C}$, defined by

$$\hat{f}(\xi) = \int_{-R}^{R} e^{-ix\xi} f(x) dx,$$

be the *Fourier transform* of f.

(a) Prove that for each fixed $\xi \in \mathbb{R}$, we have

$$e^{-ix\xi} f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (-ix\xi)^n f(x),$$

where the series on the right is uniformly convergent in x.

(b) Conclude that for every $\xi \in \mathbb{R}$,

$$\hat{f}(\xi) = \sum_{n=0}^{\infty} \left[\frac{(-i)^n}{n!} \int_{-R}^{R} x^n f(x) dx \right] \xi^n,$$

and that $\hat{f}$ is analytic in $\mathbb{R}$.

(c) Prove that if f is continuous, and if there exists $R' > 0$ such that $\hat{f}(\xi) = 0$ for $|\xi| \geq R'$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

This a simple form of the uncertainty principle: a function and its Fourier transform cannot both vanish outside of a compact set.

21. Problem 13 from page 198.