18.100B
Fall 2012

Second midterm review sheet

The second midterm covers all the material from chapters 3 – 5, and you may use results from chapters 1 – 5 without proof. Most of the midterm will consist of problems from this list. You will not be asked to reprove theorems from the book.

1. Let $A > 0$, and suppose $\sum_{n=1}^{\infty} a_n = A$, and $a_n > 0$ for all $n \in \mathbb{N}$. Find the possible values of $\sum_{n=1}^{\infty} a_n^2$.
2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Which implications are valid between the following three statements?
   (a) $(a_n)_{n\in\mathbb{N}}$ converges.
   (b) $\sum |a_{n+1} - a_n|$ converges.
   (c) $(2a_{n+1} - a_n)_{n\in\mathbb{N}}$ converges.
3. Problem 18 from page 100.
4. Problem 23 from page 101.
5. Let K be a compact metric space and $A \subset K$. Prove that A is compact if and only if for every continuous function $f\colon K \to \mathbb{R}$, the restriction of f to A attains a maximum on A. Is it possible to replace the assumption on K with a weaker one?
6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that
$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = +\infty.$$
Prove that f attains a minimum on $\mathbb{R}$.
7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, and suppose $f(x+1) = f(x)$ for all $x \in \mathbb{R}$.
   (a) Prove that f is bounded and attains both a maximum and a minimum on $\mathbb{R}$.
   (b) Prove that f is uniformly continuous on $\mathbb{R}$.
   (c) Prove that for any $a \in \mathbb{R}$ there is $x_0 \in \mathbb{R}$ such that $f(x_0+a) = f(x_0)$.
8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Prove that there exists $M > 0$ such that $|f(x)| \le M(1 + |x|)$ for all $x \in \mathbb{R}$.

9. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Prove that the set

$$\{\alpha \in \mathbb{R}\colon \exists (x_n)_{n\in\mathbb{N}} \text{ with } \lim_{n\to\infty} x_n = \infty,\ \lim_{n\to\infty} f(x_n) = \alpha\}$$

is connected.

10. Problem 2 from page 114.
11. Problem 11 from page 115.
12. Problem 14 from page 115.