18.100B
Fall 2012

Problem Set 1

Due September 13th at 4 pm in room 2-285.

Hand in parts 1 and 2 separately. Put your name on each part.

Part 1

1. Let m and n be positive integers with no common factor. Prove that if $\sqrt{m/n}$ is rational, then m and n are both perfect squares, that is to say there exist integers p and q such that $m = p^2$ and $n = q^2$. (This is proved in Proposition 9 of Book X of Euclid's *Elements*).

   You may use without proof the *Fundamental Theorem of Arithmetic*, which says that every natural number can be written as a product of primes in exactly one way, up to rearrangements.

2. Problem 8 from page 22.

3. Let $\mathbb{R}$ be the set of real numbers and suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that for all real numbers x and y the following two equations hold
$$f(x+y) = f(x) + f(y), \tag{1}$$
$$f(xy) = f(x)f(y). \tag{2}$$
   Claim: $f(x) = 0$ for all x or $f(x) = x$ for all x.
   Prove this claim using the following steps:

   (a) Prove that $f(0) = 0$ and that $f(1) = 0$ or 1.

   (b) Prove that $f(n) = nf(1)$ for every integer n and then that $f(n/m) = (n/m)f(1)$ for all integers n, m such that $m \neq 0$. Conclude that either $f(q) = 0$ for all rational numbers q or $f(q) = q$ for all rational numbers q.

   (c) Prove that f is nondecreasing, that is to say that $f(x) \geq f(y)$ whenever $x \geq y$ for any real numbers x and y.

   (d) Prove that if $f(1) = 0$ then $f(x) = 0$ for all real numbers x. Prove that if $f(1) = 1$ then $f(x) = x$ for all real numbers x.

Part 2

4. Problem 6 from page 22.
5. Problem 7 from page 22.