18.100B
Fall 2012

Problem Set 3

Due September 27th at 4 pm in room 2-285.

Hand in parts 1 and 2 separately. Put your name on each part.

Part 1

1. Problem 10 from page 44.
2. Let K be a compact metric space, and $\varepsilon > 0$. Show that there exists $N \in \mathbb{N}$ such that every set of N distinct points in K includes at least two points with distance less than ε between them.
3. Let K be a compact metric space. Show that K has a subset which is dense and at most countable.

Part 2

4. For each of the following subsets of $\mathbb{R}$, determine whether the set is open, whether it is closed, and whether it is compact. Also, find the interior, the limit points and the closure. For this problem you do not need to provide any proofs.
   (a) $\{1, 2, 3\}$.
   (b) $[-1, 0) \cup (0, 1]$.
   (c) $\mathbb{Q}$.
   (d) The complement of $\mathbb{Q}$.

   Do the same thing for the following subsets of $\mathbb{R}^2$.
   (e) $\{(x, y) \in \mathbb{R}^2 \colon y > 0\}$.
   (f) $\{(x, y) \in \mathbb{R}^2 \colon x \in [-1, 0) \cup (0, 1]\}$.
5. Let $n \in \mathbb{N}$ and let $S \subset \mathbb{R}^n$ be a set such that every point in S is isolated. Show that S is at most countable.
6. Let K be a compact metric space, and $f \colon K \to K$ a map such that $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$. Prove that there exists a point x such that $f(x) = x$. Hint: how small can $d(x, f(x))$ get?