18.100B
Fall 2012

Problem Set 5

Due October 18th at 4 pm in room 2-285.

Hand in parts 1 and 2 separately. Put your name on each part.

Part 1

1. Prove that if $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence of real numbers, then
$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup\{a_m \colon m \geq n\}\right).$$
2. Prove that a sequence in a metric space converges to a point s if and only if every subsequence has a subsequence which converges to s.

Part 2

3. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers which tends to zero but such that $\sum_{n=1}^{\infty} a_n$ diverges. Let $(A_n)_{n\in\mathbb{N}}$ be the sequence of partial sums
$$A_n = \sum_{k=1}^{n} a_k,$$
and let $b_{n+1} = \sqrt{A_{n+1}} - \sqrt{A_n}$. Show that
$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0,$$
but that $\sum_{n=1}^{\infty} b_n$ is still divergent. In this sense there is no 'smallest' divergent series, and one can similarly show that there is no 'largest' convergent one.
4. Problem 24 from page 82.