18.100B
Fall 2012

Problem Set 6

Due October 25th at 4 pm in room 2-285.

Hand in parts 1 and 2 separately. Put your name on each part.

Part 1

1. Problem 6 from page 78.
2. Suppose $(a_n)_{n\in\mathbb{N}}$ satisfies $a_{n+m} \leq a_n + a_m$ for all $m, n \in \mathbb{N}$. Prove that
$$\lim_{n\to\infty} \frac{a_n}{n} = \inf\left\{\frac{a_n}{n} : n \in \mathbb{N}\right\},$$
as an element of $\mathbb{R} \cup \{-\infty\}$. You may use without proof the *Euclidean division algorithm*, which says that for any $n, \ell \in \mathbb{N}$, there exist unique $m, r \in \mathbb{N} \cup \{0\}$ with $r < \ell$ such that
$$n = m\ell + r.$$
3. Let $\sum b_n$ be a convergent series of real numbers, and let $(a_n)_{n\in\mathbb{N}}$ be bounded below. Prove that if
$$a_{n+1} \leq a_n + b_n, \qquad \forall n \in \mathbb{N},$$
then $(a_n)_{n\in\mathbb{N}}$ converges.

Part 2

4. Let X and Y be metric spaces and $f \colon X \to Y$ a function. Prove that the following are equivalent.
   (a) f is continuous.
   (b) If $(p_n)_{n\in\mathbb{N}}$ converges in X, then $(f(p_n))_{n\in\mathbb{N}}$ converges in Y.
   (c) If $(p_n)_{n\in\mathbb{N}}$ converges in X, then $(f(p_n))_{n\in\mathbb{N}}$ converges in Y and
   $$\lim_{n\to\infty} f(p_n) = f\left(\lim_{n\to\infty} p_n\right).$$
   (d) $f(\overline{E}) \subset \overline{f(E)}$ for any $E \subset X$.
5. Problem 4 from page 98.