18.100B
Fall 2012

Problem Set 9

Due November 29th at 4 pm in room 2-285.

Hand in parts 1 and 2 separately. Put your name on each part.

Part 1

1. Let $x > 0$, let $n \in \mathbb{N} \cup \{0\}$, and let $f\colon [0, x] \to \mathbb{R}$ be $n + 1$ times differentiable with $f^{(n+1)}$ integrable. Use mathematical induction and integration by parts to prove that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + I_n(x),$$

where

$$I_n(x) = \frac{x^{n+1}}{n!}\int_0^1 (1-t)^n f^{(n+1)}(tx)\,dt = \frac{1}{n!}\int_0^x (x - x')^n f^{(n+1)}(x')\,dx'.$$

Theorem 5.15 in Rudin uses the mean value theorem to prove another version of Taylor's theorem under slightly weaker hypotheses, but this version has the advantage of giving a more explicit remainder. Of course the case $x < 0$ follows by applying the result to $g(x) = f(-x)$, and an expansion near $a \neq 0$ follows by taking $g(x) = f(a + x)$.

2. Problem 2 from page 165.
3. Problem 3 from page 165.

Part 2

4. Problem 16 from page 168.
5. Problem 24 from page 170.