18.104
Spring 2013

Homework 12

Due April 11th by the beginning of class.

Problem: For $\lambda \in \mathbb{C}$ with Im $\lambda > 0$, put

$$f(x) = (x^2 - \lambda^2)^{-1}.$$

Prove that

$$\hat{f}(\xi) = \frac{i\pi}{\lambda} e^{i\lambda|\xi|},$$

and use the Poisson summation formula [FrJo, (8.5.3)] to show that

$$\sum_{k=-\infty}^{\infty} f(k) = \frac{\pi}{\lambda} \cot(\pi\lambda).$$

Hint: If you don't know the residue theorem from complex analysis, it might be easier to calculate the Fourier transform of $e^{i\lambda|x|}$ and use the Fourier inversion formula. The sum that results from Poisson summation reduces to a geometric series.

Hint 2: The Poisson summation formula [FrJo, (8.5.3)] implies that if $f \in \mathscr{S}(\mathbb{R})$, then $\sum_{k=-\infty}^{+\infty} f(k) = \sum_{k=-\infty}^{+\infty} \hat{f}(2\pi k)$. In this problem you want to use this equation for a more general f. In fact, the equation is true as long as $f \in \mathscr{S}'(\mathbb{R})$ is such that both f and $\hat{f}$ are continuous and there are constants $C > 0$ and $\delta > 0$ such that $|f(t)| + |\hat{f}(t)| \leq C(1+|t|)^{-1-\delta}$ for all t in $\mathbb{R}$. For this problem you may use this version of Poisson summation without proof.

Solution:

References

[FrJo] G. Friedlander and M. Joshi. The Theory of Distributions, second edition, Cambridge University Press, 1998.