# Final paper

The final paper must be about 10 pages long, written in Latex, and on a topic related to the course; it must also connect the material in the book with something in another source or sources. The topic must be chosen by March 21st, a first draft (which must be complete) is due April 16th, and the final draft is due April 30th. Here is a list of suggested topics, with recommended starting points. Let me know if you are interested in something not on this list. I am also very happy to meet with you for discussions both before and after you choose your topic.

I prefer for each student to have a different topic, so please send me an email by March 21st dividing the topics into three categories: topics you would be excited about, topics you don't mind, and topics you'd rather not work on. I will let you know which topic you've been assigned by the following day.

Also include in your email a ranking of the following presentation dates from most to least preferred: 4/30, 5/2, 5/7, 5/9, 5/14, 5/16.

(1) Smooth functions and their derivatives. Borel's Theorem says that for any sequence $(a_j)_{j=0}^\infty$ of complex numbers, there is $f \in C^\infty(\mathbb{R})$ such that $\partial^j f(0) = a_j$ (see e.g. [Hö2, Theorem 1.2.6]). On the other hand, if $f \in C_c^\infty(\mathbb{R})$ but $f \not\equiv 0$, then the sequence $\sup |\partial^j f|$ cannot grow too slowly: see [Hö2, Theorem 1.3.5 and Lemma 1.3.6].

(2) The topology of $\mathscr{D}'$. The theory of topological vector spaces applied to $\mathscr{D}'$ can be used to give a proof of [FrJo, Theorem 1.5.2] by the Banach–Steinhaus uniform boundedness principle (see [Do, §20] or [Ru, Theorem 6.17]).

(3) Distributions on manifolds. The behavior of distributions under coordinate transformations and pullbacks (see [FrJo, Chapter 7]) leads naturally to a theory of distributions on manifolds. One place to read about this is [Hö, §1.8].

(4) The Schwartz kernel theorem [FrJo, Chapter 6] identifies sequentially continuous operators $P \colon C_c^\infty(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ with $\mathscr{D}'(\mathbb{R}^{2n})$. This can be applied, for example, to Peetre's theorem [Pe], which says that if $\operatorname{supp} Pu \subset \operatorname{supp} u$ for all $u \in C_c^\infty(\mathbb{R}^n)$ then $P$ is necessarily a differential operator (see also e.g. [Gr]).

(5) Calderón's Inverse Problem asks: when can the electrical conductivity of an object be recovered from boundary measurements of voltage and current? A good starting point is Calderón's original paper [Ca]; see also e.g. [Sa, §3].

(6) The Malgrange–Ehrenpreis Theorem says that any constant coefficient linear partial differential operator has a fundamental solution, and hence that any constant coefficient PDE can be solved. This has many proofs, giving more or less explicit formulas and fundamental solutions with various desirable properties. One is in [FrJo, §10.4], another in [Wa], and a local solvability proof is in [Ta, §1.7].

(7) Smooth linear partial differential equations, in contrast to ones with constant coefficients, do not always have solutions, even locally. This was first shown by Hans Lewy in [Le]; you might also consult [Fo, §1.E] and [Jo, Chapter 8].

(8) A differential operator $P$ is *hypoelliptic* if $Pu \in C^\infty$ implies $u \in C^\infty$. For example the Laplace operator $\Delta = \sum_{j=1}^n \partial_j^2$ is hypoelliptic but the wave operator $\partial_t^2 - \Delta$ is not.

Hörmander's theorem characterizes the constant coefficient hypoelliptic operators algebraically: see for example [Fo, §6.D] or [Ta, §3.2].

(9) Pseudodifferential operators. A linear differential operator is a 'function' $P(x, \partial)$ which is a polynomial in $\partial$: a pseudodifferential operator is one where $P$ is replaced by a more general function. One consequence of their theory is a generalization of the elliptic regularity theorem [FrJo, Theorem 8.6.1] to operators with nonconstant coefficients (see e.g. [Ta, §3.1]).

(10) Morawetz estimates for the Schrödinger equation $i\partial_t u + \Delta u = 0$ give constraints on the degree to which a quantum wave function can concentrate near any given point, see e.g. [Wu, §2.1].

(11) The Radon transform of a function is obtained by considering its integral along lines or hyperplanes and has a nice inversion formula involving fractional powers of the Laplace operator, see e.g. §5.2 and Problem 8 of [StSh, Chapter 6].

(12) The Poisson summation formula, which we will see in [FrJo, §8.5] can be used to prove Minkowski's theorem on lattice points in a convex body. The theory of the 'geometry of numbers' which this leads to has applications in number theory (see e.g. [Do, §34] and [Kö, §79]).

(13) Another application of the Poisson summation formula is to study asymptotics of the number of lattice points in a ball (see e.g. Landau's asymptotic formula [Pi, §4.5]).

(14) Fundamental solutions of the wave equation. These can be obtained as another application of the behavior of distributions under coordinate transformations and pullbacks (see [FrJo, §7.3], [FrJo, Exercise 7.4], and [Fr, §6.1]).

(15) The Fourier transform, combined with a Tauberian theorem for series convergence, leads to a proof of the prime number theorem, which gives an asymptotic formula for the number of primes less than or equal to a given integer (see e.g. [Ru, pp.208–217]).

## References


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