MA 341 supplement to Mattuck's *Introduction to Analysis*

I will try to add to this as the semester goes on. Please send any questions, comments, or corrections to kdatchev@purdue.edu.

Section 2.4

PROBLEM. Give an upper estimate, in terms of n alone, for

$$\frac{3}{4+n^2+a^2+\cos(b2^n)}.$$

Show that this estimate is the best possible, by giving values of a and b for which the bound is actually attained.

SOLUTION. Combining $0 \le a^2$ and $-1 \le \cos(b2^n)$ gives

$$3+n^2 \le 4+n^2+a^2+\cos(b2^n).$$

Consequently,

$$\frac{3}{4+n^2+a^2+\cos(b2^n)} \le \frac{3}{3+n^2}.$$

To see that this upper estimate is the best possible we observe that the bound is attained when $0 = a^2$ and $-1 = \cos(b2^n)$. This occurs when $a = 0$ and $b2^n = \pi$, i.e. when $a = 0$ and $b = \pi 2^{-n}$.

Section 2.5

PROBLEM. Find a number c such that if $|a - \frac{1}{2}| < \varepsilon < \frac{1}{4}$, then $|a^{-1} - 2| < c\varepsilon$.

SOLUTION. From $|a - \frac{1}{2}| < \frac{1}{4}$ we have $-\frac{1}{4} < a - \frac{1}{2} < \frac{1}{4}$ or $\frac{1}{4} < a < \frac{3}{4}$ and so $\frac{4}{3} < \frac{1}{a} < 4$. Next

$$|a^{-1} - 2| = \left|\frac{1-2a}{a}\right| = \frac{2}{a}\left|\frac{1}{2} - a\right| < \frac{2}{a}\varepsilon < 8\varepsilon.$$

Thus we may take $c = 8$.

Section 3.1

PROBLEM. Find the limit L of the sequence

$$a_n = \frac{3n+100}{n+1}.$$

Find a number N such that $|a_n - L| < 0.1$ when $n > N$.

SOLUTION. We have $L = 3$ because the dominant terms in the numerator and denominator are $3n$ and n respectively. To prove this from the definition, given $\varepsilon > 0$ we must find N such that $|a_n - 3| < \varepsilon$ when $n > N$. We write

$$|a_n - 3| = \frac{3n+100-3n-3}{n+1} = \frac{97}{n+1}.$$

This is $< \varepsilon$ when $n+1 > 97/\varepsilon$, or $n > -1 + 97/\varepsilon$, so we can take $N = -1 + 97/\varepsilon$. If $\varepsilon = 0.1$, then we may take $N = 969$.

Section 6.2

PROBLEM. Let

$$a_n = \frac{1}{n} + \sin\left(\frac{2n-1}{4}\pi\right).$$

Find two cluster points of the sequence $a_1,\ a_2, \ldots$ and, for each cluster point, find a subsequence converging to it.

SOLUTION. If $n = 4k$, then

$$a_{4k} = \frac{1}{4k} + \sin\left(\frac{8k-1}{4}\pi\right) = \frac{1}{4k} + \sin\left(\frac{-1}{4}\pi\right) \to \sin\left(\frac{-1}{4}\pi\right) = \frac{-1}{\sqrt{2}}.$$

If $n = 4k+1$, then

$$a_{4k+1} = \frac{1}{4k+1} + \sin\left(\frac{8k+1}{4}\pi\right) = \frac{1}{4k+1} + \sin\left(\frac{1}{4}\pi\right) \to \sin\left(\frac{1}{4}\pi\right) = \frac{1}{\sqrt{2}}.$$

So $1/\sqrt{2}$ and $-1/\sqrt{2}$ are cluster points, and the subsequences a_{4k} and a_{4k+1} converge to them respectively.

Section 6.5

PROBLEM. Let

$$S = \{\frac{\cos 1}{1}, \frac{\cos 2}{2}, \frac{\cos 3}{3}, \ldots\}.$$

Use the inequalities $\cos 1 > .5$ and $-.5 < \cos 2$ and $\cos 3 < -.9$ to find $\sup S$ and $\inf S$.

SOLUTION.

If $n \geq 4$, then $-1 \leq \cos n$ implies $-.25 \leq (\cos n)/n$. Since $.5 < \cos 1$ and $-.25 < (\cos 2)/2$ it follows that $(\cos 3)/3 < -.3 < (\cos n)/n$ when $n \neq 3$. Hence $\inf S = \cos 3$.

Similarly, if $n \geq 4$, then $\cos n \leq 1$ implies $(\cos n)/n \leq .25$. Using $(\cos 2)/2 < .25$ and $(\cos 3)/3 < -.3$ shows that $(\cos n)/n < .5 < \cos 1$ when $n \neq 1$. Hence $\sup S = \cos 1$.

Section 7.4

PROBLEM. Let

$$a_n = \frac{100^n}{n!}$$

Prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, and use this result to find $\displaystyle\lim_{n\to\infty} a_n$.

SOLUTION. We have

$$\frac{a_{n+1}}{a_n} = \frac{100^{n+1}}{(n+1)!}\frac{n!}{100^n} = \frac{100}{n+1} \to 0,$$

so the series converges by the ratio test. The terms of a convergent series tend to zero, so $\displaystyle\lim_{n\to\infty} a_n = 0$.

Section 11.1

Problem. Let $f(x)$ defined by

$$f(x) = x^{1/7}\cos(1/x), \qquad \text{when } x \neq 0,$$

and $f(0) = 0$. Prove directly from the definition that f is continuous at 0.

Solution. We must check that for any given $\varepsilon > 0$ there is $\delta > 0$ such that $|x| < \delta$ implies $|f(x)| < \varepsilon$. If $x = 0$, then $f(x) = 0$ and there is nothing further to check. If $x \neq 0$ and $|x| < \delta$, then

$$|f(x)| = |x|^{1/7}|\cos(1/x)| < \delta^{1/7},$$

since $|\cos(1/x)| \leq 1$. We may set $\delta^{1/7} = \varepsilon$, or $\delta = \varepsilon^7$. Any smaller choice would also work, such as $\delta = \varepsilon^7/2$ or $\delta = \varepsilon^7/100$.

Problem.

Let $f(x) = 2x + 3$. Given $\varepsilon > 0$, find $\delta > 0$ such that $|f(x) - 11| < \varepsilon$ when $|x - 4| < \delta$.

Solution.

If $|x - 4| < \delta$, then $|f(x) - 11| = |2x + 3 - 11| = |2x - 8| = 2|x - 4| < 2\delta$. Thus we may set $2\delta = \varepsilon$ to obtain $|f(x) - 11| < \varepsilon$. In other words, one solution is $\delta = \varepsilon/2$.

Section 12.2

Problem. Give a lower estimate of the form $aA + b$ with $a > 0$ for the number of solutions to $\sin x = x/A$, when A is large.

Solution. Start by counting solutions on intervals of the form $[2\pi n, 2\pi n + 2\pi)$, where $n \geq 0$. Because $|\sin x| \leq 1$, we need to have $|x/A| \leq 1$, or $x \leq A$. Since we just need a lower bound we only look at intervals such that

$$2\pi n + 2\pi \leq A. \tag{1}$$

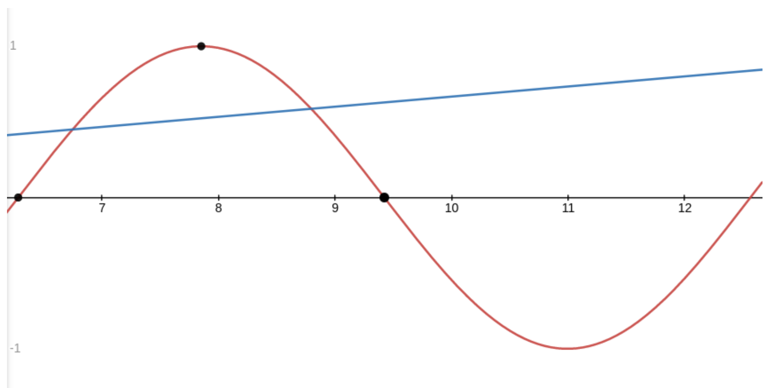

Figure 1. We use points x where $\sin x = 0$ or 1 as endpoints for the intervals on which we apply the intermediate value theorem.

We apply the intermediate value theorem to the function

$$f(x) = \sin x - \frac{x}{A}.$$

We have

$$f(2\pi n) = -\frac{2\pi n}{A} < 0,$$

and

$$f\big(2\pi n + \frac{\pi}{2}\big) = 1 - \frac{2\pi n + \frac{\pi}{2}}{A} = \frac{A - 2\pi n - \frac{\pi}{2}}{A} > 0,$$

and

$$f(2\pi n + \pi) = -\frac{2\pi n + \pi}{A} < 0.$$

hence, by the intermediate value theorem, $f(x) = 0$ has a solution in $(2\pi n, 2\pi n + \frac{\pi}{2})$ and a solution in $(2\pi n + \frac{\pi}{2}, 2\pi n + \pi)$. To determine the number of intervals such that (1) holds, we fix a number N such that $A \in [2\pi N, 2\pi N + 2\pi)$, i.e.

$$2\pi N \leq A \leq 2\pi N + 2\pi. \tag{2}$$

Then we have (1) for $n = 0, 1, \ldots, N-1$, and so we have found $2N$ solutions in $[0, 2\pi N)$. Since $-x$ is a solution whenever x is, we have at least $4N - 1$ real solutions (avoiding double-counting the solution $x = 0$ and ignoring a few possible solutions with $|x| \geq 2\pi N$). Using (2), we have at least

$$4N - 1 \geq 4\big(\frac{A}{2\pi} - 1\big) - 1 = \frac{2}{\pi}A - 5$$

solutions overall. Thus we have the desired lower estimate with $a = 2/\pi$ and $b = -5$.

Section 13.2

Problem. Let $f\colon [2,5) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to 5^-} = 4$. Prove that f is bounded on $[2,5)$.

Solution. By the boundedness theorem, we know that f is bounded on $[2,c]$ for any $c \in (2,5)$. We have to choose c in such a way that f is also bounded on $(c,5)$.

To do this, use the limit definition. There is $\delta > 0$ such that $|f(x) - 4| < 1$ when $5 - \delta < x < 5$. That means $3 < f(x) < 5$ on $(5-\delta, 5)$.

Thus, by the boundedness theorem, there are numbers b_1 and b_2 such that $b_1 \le f(x) \le b_2$ for all $x \in [2, 5-\delta_1]$. But we also have $3 < f(x) < 5$ on $(5-\delta, 5)$. Combining, we have

$$\min(b_1, 3) \le f(x) \le \max(b_2, 5)$$

for all $x \in [2,5)$.

Section 13.3

Problem. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = +\infty$$

Show that f has a minimum on $\mathbb{R}$.

Solution. By the maximum theorem, for any $[a,b]$ there is $c \in [a,b]$ such that $f(c) \le f(x)$ for all $x \in [a,b]$. We need to chose $[a,b]$ in such a ways that $f(c) \le f(x)$ for all x outside of $[a,b]$ as well.

To do this, use the limit definition. There is N_1 such that $x \ge N_1$ implies $f(x) \ge f(0)$. Similarly, there is N_2 such that $f(x) \ge f(0)$ when $x \le N_2$.

Use $N_2 = a$ and $N_1 = b$. Then, by the maximum theorem, there is c such that $f(c) \le f(x)$ for all $x \in [a,b]$. We also have $f(c) \le f(0) \le f(x)$ for all x not in $[a,b]$.

Problem. Let $f\colon (0,\infty) \to \mathbb{R}$ be continuous and such that $\lim_{x\to\infty} f(x) = \infty$. Prove that f is bounded below on $(1,\infty)$. Prove that f has a minimum on $[2,\infty)$. Give an example of f satisfying the above properties such that f has no minimum on $(0,\infty)$.

Solution. By the definition of $\lim_{x\to\infty} f(x) = \infty$, there is N such that $f(x) > 1$ when $x > N$. By the boundedness theorem, there is b such that $f(x) > b$ when $x \in [1,N]$. Hence $x \ge \min(b,1)$ when $x > 1$ and so f is bounded below on $(1,\infty)$.

Similarly, by the definition of $\lim_{x\to\infty} f(x) = \infty$, there is M such that $f(x) > f(2)$ when $x > M$. If $2 \ge M$ then we have $f(x) \ge f(2)$ when $x \ge 2$ and we are done. If $M > 2$, then, by the minimum theorem, there is $c \in [2,M]$ such that $f(x) \ge f(c)$ for all $x \in [2,M]$. Since $f(x) > f(2) \ge f(c)$ for all $x > M$, it follows that $f(x) \ge f(c)$ for all $x \in [2,\infty)$.

An example of f satisfying the above properties with no minimum on $(0,\infty)$ is $f(x) = x$. It has no minimum on $(0,\infty)$ because for every $c > 0$, we have $f(x) < f(c)$ when $x \in (0,c)$, so $f(c)$ cannot be the minimum.

Section 15.1

PROBLEM. Let a be a given real number, and let $n \geq 1$ be an integer. Find all real solutions to $x^n + a^n = (x+a)^n$.

SOLUTION. If $a = 0$ this becomes $x^n = x^n$ and all real x are solutions. If $n = 1$ it becomes $x + a = x + a$ and all real x are solutions.

To analyze other values of a and n, let $f(x) = x^n + a^n - (x+a)^n$, and we are solving $f(x) = 0$. By Rolle's theorem, there is a zero of f' between any two zeroes of f. We have $f'(x) = nx^{n-1} - n(x+a)^{n-1}$, and this is zero if and only if $x^{n-1} = (x+a)^{n-1}$.

If n is even and $a \neq 0$, then $x^{n-1} = (x+a)^{n-1}$ is equivalent to $x = x + a$ which has no solutions, so $f(x) = 0$ has at most one solution. Since $f(0) = 0$, we see that $x = 0$ is the only solution to $x^n + a^n = (x+a)^n$ then.

If $n \geq 3$ is odd and $a \neq 0$, then $x^{n-1} = (x+a)^{n-1}$ has one solution, namely $x = -(x+a)$, or $x = -a/2$. That means $f(x) = 0$ has at most two solutions. We check that $f(0) = 0$ and $f(-a) = 0$, so $x = 0$ and $x = -a$ are the only solutions to $x^n + a^n = (x+a)^n$ then.

In summary, if $a = 0$ or $n = 1$, then all real x are solutions. If $a \neq 0$ and n is even, then $x = 0$ is the only solution. If $a \neq 0$ and $n \geq 3$ is odd, then $x = 0$ and $x = -a$ are the only solutions.

Section 19.4

PROBLEM. Find numbers a and b such that for all real c we have

$$0 < a \leq \int_0^1 \frac{x}{2 + \sin cx} dx \leq b < 1.$$

SOLUTION. From $-1 \leq \sin cx \leq 1$ we get $1 \leq 2 + \sin cx \leq 3$ and hence $\frac{1}{3} \leq \frac{1}{2+\sin cx} \leq 1$ and $\frac{x}{3} \leq \frac{x}{2+\sin cx} \leq x$. Since $\int_0^1 \frac{x}{3} dx = \frac{1}{6}$ and $\int_0^1 x dx = \frac{1}{2}$ we may take $a = 1/6$ and $b = 1/2$.

Section 20.4

Let us derive the series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \tag{3}$$

from $\exp'(x) = x$ and $\exp(0) = 1$. Recall that the series on the right converges for all x by the ratio test. Deriving (3) also shows that setting $e = \exp(1)$ is consistent with our previous definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

From the fundamental theorem of calculus we get

$$\int_0^x \exp(t)dt = \exp(x) - \exp(0) = \exp(x) - 1,$$

and hence

$$\exp(x) = 1 + \int_0^x \exp(t)dt. \tag{4}$$

Plugging (4) into itself, i.e. plugging $\exp(t) = 1 + \int_0^t \exp(t_1)dt_1$ into (4), gives

$$\begin{aligned}\exp(x) &= 1 + \int_0^x \Big(1 + \int_0^t \exp(t_1)dt_1\Big)dt \\ &= 1 + x + \int_0^x \int_0^t \exp(t_1)dt_1 dt,\end{aligned}$$

and we have the first two terms of the series (3).

Plugging in (4) again, this time in the form $\exp(t_1) = 1 + \int_0^{t_1} \exp(t_2)dt_2$, gives

$$\begin{aligned}\exp(x) &= 1 + x + \int_0^x \int_0^t \Big(1 + \int_0^{t_1} \exp(t_2)dt_2\Big)dt_1 dt \\ &= 1 + x + \frac{x^2}{2} + \int_0^x \int_0^t \int_0^{t_1} \exp(t_2)dt_2 dt_1 dt.\end{aligned}$$

Continuing in this manner we obtain, for any n,

$$\exp(x) = 1 + x + \cdots + \frac{x^n}{n!} + \int_0^x \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \exp(t_n)dt_n \cdots dt_1 dt.$$

To conclude, we must show that the remainders go to zero as $n \to \infty$. When $x \geq 0$, we have

$$\begin{aligned}\Big|\int_0^x \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \exp(t_n)dt_n \cdots dt_1 dt\Big| &\leq \exp(x) \int_0^x \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \cdots dt_1 dt \\ &= \exp(x) \frac{x^{n+1}}{(n+1)!}\end{aligned}$$

And when $x \leq 0$, we have

$$\begin{aligned}\Big|\int_0^x \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \exp(t_n)dt_n \cdots dt_1 dt\Big| &\leq \int_0^x \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \cdots dt_1 dt \\ &= \frac{x^{n+1}}{(n+1)!}\end{aligned}$$

In both cases the right hand side goes to zero as $n \to \infty$, because it is the n-th term of a convergent series.