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MA 341
Fall 2021

Homework 5

Due Friday, October 15th at the beginning of class. Note the unusual day of the week. Justify your answers. Please let me know if you have a question or find a mistake.

1. Let
$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 2, & x = 0. \end{cases}$$
Use the estimate
$$|\sin x - x| \leq |x^3|/6,$$
to find $L = \lim_{x\to 0} f(x)$, and also find a $\delta > 0$ such that
$$0 \leq L - f(x) \leq \frac{2}{27}, \qquad \text{whenever } 0 < |x| < \delta.$$

2. Exercise 11.1.6 from page 167. Instead of using the hint in the book, you can use the alternating series
$$\cos a = 1 - \frac{a^2}{2} + \frac{a^4}{4!} - \ldots.$$

3. Exercise 12.1.5 from page 180.

4. Use Theorem 12.1 to find three intervals of length 1, each one containing a solution to
$$3^x = 4x^2.$$

5. Let $a_0 < b_0$ and a continuous function $f \colon [a_0, b_0] \to \mathbb{R}$ be given. Show that f is bounded on $[a_0, b_0]$ by following the steps below.
   (a) Suppose for contradiction that f is unbounded on $[a_0, b_0]$. Prove that f is unbounded on at least one of the subintervals $[a_0, (a_0 + b_0)/2]$ and $[(a_0 + b_0)/2, b_0]$.

   Hint: Argue by contrapositive, showing that if f is bounded on both subintervals then it is bounded on the whole thing.

   (b) Use repeated bisection (as in the proofs of Theorem 6.3, Theorem 6.5A, and Theorem 12.1) to define a sequence of nested intervals $[a_0, b_0], [a_1, b_1], \ldots$ such that f is unbounded on all of them. Apply the Nested Intervals Theorem (Theorem 6.1) to show there is a point c in all the intervals.

   (c) Use the fact that f is continuous at c to show there is an interval $(c - \delta, c + \delta)$ for some $\delta > 0$ such that f is bounded on $(c - \delta, c + \delta)$.

   (d) Find n, in terms of δ, such that the interval $[a_n, b_n]$ is contained in the interval $(c-\delta, c+\delta)$, and obtain a contradiction by combining with what you found in parts (b) and (c).